

Generic coverings of plane with A-D-E-singularities.

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Abstract

We investigate a presentation of an algebraic surface X with A-D-E-singularities as a generic covering $f : X \rightarrow \mathbb{P}^2$, i.e. a finite morphism, having at most folds and pleats apart from singular points, isomorphic to a projection of a surface $z^2 = h(x, y)$ onto the plane x, y in neighbourhoods of singular points, and the branch curve $B \subset \mathbb{P}^2$ of which has only nodes and ordinary cusps except singularities originated from the singularities of X . It is deemed that classics proved that a generic projection of a non-singular surface $X \subset \mathbb{P}^r$ is of such form. In this paper this result is proved for an embedding of a surface X with A-D-E-singularities, which is a composition of the given one and a Veronese embedding. We generalize results of the paper [K], in which Chisini's conjecture on the unique reconstruction of f by the curve B is investigated. For this fibre products of generic coverings are studied. The main inequality bounding the degree of a covering in the case of existence of two nonequivalent coverings with the branch curve B is obtained. This inequality is used for the proof of the Chisini conjecture for m -canonical coverings of surfaces of general type for $m \geq 5$.

Introduction

Let $S \subset \mathbb{P}^r$ be a non-singular projective surface, $f : S \rightarrow \mathbb{P}^2$ be its generic projection to the plane, $B \subset \mathbb{P}^2$ be the branch curve, which we call the discriminant curve. It is deemed that classics proved (see [Z], p.104) that (i) the map f is a finite covering, which has as singularities at most double points (folds), or singular points of cuspidal type (pleats); (ii) with this $f^*(B) = 2R + C$, where the double curve R is non-singular and irreducible, and the curve C is reduced; (iii) the curve B is cuspidal, i.e. has at most nodes and ordinary cusps; over a node there lie two double points, and over a cusp – one point of cuspidal type; (iv) the restriction of f to R is of degree one. Any finite morphism $f : S \rightarrow \mathbb{P}^2$ is called a *generic* (or *simple*) *covering*, if it possesses the same properties as a generic projection. Two coverings of plane (S_1, f_1) and (S_2, f_2) are called equivalent, if there is a morphism $\varphi : S_1 \rightarrow S_2$ such that $f_1 = f_2 \circ \varphi$.

In this paper we consider a generalization of the notion of a generic covering to the case of surfaces with A-D-E-singularities. First of all we want to explain why we need such a generalization. A presentation of an algebraic variety as a finite covering of the projective space is one of the effective ways of studying projective varieties as well as their moduli. To

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compare we recall what such an approach gives in the case of curves. For a curve C of genus g a generic covering $f : C \rightarrow \mathbb{P}^1$ is such a covering that in every fibre there is at most one ramification point which is a double point (or a singular point of f of type A_1). Let $B \subset \mathbb{P}^1$ be the set of branch points, and $d = \deg B$, i.e. $d = \sharp(B)$. Then according to the Hurwitz formula $d = 2N + 2g - 2$, where $N = \deg f$. If $N \geq g + 1$, then any curve of genus g can be presented as a simple covering of \mathbb{P}^1 of degree N . The set of all simple coverings (up to equivalence) $f : C \rightarrow \mathbb{P}^1$ of degree N with d branch points is parametrized by a Hurwitz variety $H = H^{N,d}$. Let $\mathbb{P}^d \setminus \Delta$ (Δ – discriminant) be the projective space parametrizing the sets of d different points of \mathbb{P}^1 , and let M_g be the moduli space of curves of genus g . There are two maps: a map $h : H \rightarrow \mathbb{P}^d \setminus \Delta$, sending f to the set of branch points $B \subset \mathbb{P}^1$, and a map $\mu : H \rightarrow M_g$, sending f to the class of curves isomorphic to C . Hurwitz introduced and investigated the variety H in 1891. He proved that the variety H is connected, and h is a finite unramified covering. In modern functorial language H was studied also by W. Fulton in 1969. The map μ is surjective (and has fibres of dimension $N + (N - g + 1)$). This gives one of the proofs of irreducibility of the moduli space M_g .

In the case of surfaces we also can consider an analog of Hurwitz variety H of all generic coverings (up to equivalence) $f : S \rightarrow \mathbb{P}^2$ of degree N and with discriminant curve B of degree d with given number n of nodes and given number c of cusps. Let \mathbb{P}^ν , $\nu = \frac{d(d+3)}{2}$, be a projective space parametrizing curves of degree d , and $h : H \rightarrow \mathbb{P}^\nu$ be a map sending a covering f to its discriminant curve B . In [K] a Chisini conjecture is studied. It claims that if B is the discriminant curve of a generic covering f of degree $N \geq 5$, then f is uniquely up to equivalence defined by the curve B . In other words, it means that the map h is injective (and, besides, $N = \deg f$ is determined by B). In [K] it is proved that the Chisini conjecture is true for almost all generic coverings. In particular, it is true for generic coverings defined by a multiple canonical class. A construction of the moduli space of surfaces of general type uses pluricanonical maps. As is known [BPV], if S is a minimal surface of general type, then for $m \geq 5$ the linear system $|mK_S|$ blows down only (-2) -curves and gives a birational map of S to a surface $X \subset \mathbb{P}^r$ (the canonical model) with at most A-D-E-singularities (in other terms, rational double points, Du Val singularities, simple singularities of Arnol'd and etc.). This requires a generalization of the notion of a generic covering to the case of surfaces with A-D-E-singularities.

In this paper we, firstly, generalize a classical result on singularities of generic projections of non-singular surfaces to the case of surfaces with A-D-E-singularities. We prove that if a surface $X \subset \mathbb{P}^r$ has at most A-D-E-singularities, then (may be after a "twist") for a generic projection $f : X \rightarrow \mathbb{P}^2$ the discriminant curve B also has at most A-D-E-singularities. It follows from a slightly more general theorem.

Theorem 0.1 *Let $X \subset \mathbb{P}^r$ be a surface with at most isolated singularities of the form $z^2 = h(x, y)$ (= "double planes"), $X \rightarrow \mathbb{P}^2$ be the restriction to X of a generic projection $\mathbb{P}^r \setminus L \rightarrow \mathbb{P}^2$ from a generic linear subspace L of dimension $r - 3$. Then*

- (i) *f is a finite covering;*
- (ii) *at non-singular points of X the covering f has as singularities at most either double points (folds), or singular points of cuspidal type (pleats); in a neighbourhood of these points f*

is equivalent to a projection of a surface $x = z^2$, respectively $y = z^3 + xz$, to the plane x, y ;

(iii) in a neighbourhood of a point $s \in \text{Sing } X$ the covering f is analytically equivalent to a projection of a surface $z^2 = h(x, y)$ to the plane x, y ; from (ii) and (iii) it follows that the ramification divisor is reduced, i.e. $f^*(B) = 2R + C$, where $B = f(R)$, and R and C are reduced curves;

(iv) except singular points $f(\text{Sing } X)$ the discriminant curve B is cuspidal;

(v) the restriction of f to R is of degree 1.

Actually, the main difficulty in the proof of this theorem lies in the classical case, when the surface X is non-singular. Unfortunately, authors do not know a complete (and modern) proof of this theorem, and it seems that such a proof does not exist. Thus, the proof, even in the case of a non-singular surface, take interest. In this paper we prove a weakened version of Theorem 0.1, in which the initial embedding is ‘twisted’ by a Veronese embedding. This is quite enough for the purposes described above.

Thus, the curve B has, firstly, ‘the same’ singularities as the surface X (and as the curve R), which are locally defined by the equation $h(x, y) = 0$. These singularities on B we call *s-singularities*, in particular, *s-nodes* and *s-cusps*. Besides, there are nodes and cusps on B originated from singularities of the map f , which we call *p-nodes* and *p-cusps*. There are two double points of f over a p-node, at which f is defined locally as a projection of surfaces $z_1 = x^2$ and $z_2 = y^2$ to the plane x, y .

If S is a surface with A-D-E-singularities, then a covering $f : S \rightarrow \mathbb{P}^2$ is called *generic*, if it satisfies the properties of Theorem 0.1.

Secondly, we generalize the central result of [K] to the case of surfaces with A-D-E-singularities. It is proved there that if a generic covering $f : S \rightarrow \mathbb{P}^2$ of a non-singular surface S with discriminant curve B is of sufficiently big degree $\deg f = N$, namely under condition

$$N > \frac{4(3\bar{d} + g - 1)}{2(3\bar{d} + g - 1) - c}, \quad (1)$$

where $2\bar{d} = \deg B$, g be the geometric genus of B , and c be the number of cusps, then B is the discriminant curve of a unique generic covering (the Chisini conjecture holds for B).

We can’t expect an analogous result in the case of singular surfaces, because for a curve B of even degree with at most A-D-E-singularities there always exists a double covering, which is generic. But if two generic coverings with given discriminant curve B are coverings of sufficiently big degree, then they are equivalent. More exactly, we prove the following theorem. Let there are two generic coverings $f_1 : X_1 \rightarrow \mathbb{P}^2$ and $f_2 : X_2 \rightarrow \mathbb{P}^2$ of surfaces with A-D-E-singularities and with the same discriminant curve $B \subset \mathbb{P}^2$. Let $f_i^*(B) = 2R_i + C_i$, $i = 1, 2$. With respect to a pair of coverings f_1 and f_2 nodes and cusps of B are partitioned into four types: ss-, sp-, ps- and pp-nodes and cusps. For example, a sp-node $b \in B$ is a node, which is a s-node for f_1 and a p-node for f_2 . The number of sp-nodes is denoted by n_{sp} . Then $n = n_{ss} + n_{sp} + n_{ps} + n_{pp}$. The analogous terminology is used for cusps.

Theorem 0.2 *If f_1 and f_2 are nonequivalent generic coverings, then*

$$\deg f_2 \leq \frac{4(3\bar{d} + g_1 - 1)}{2(3\bar{d} + g_1 - 1) - \iota_1}, \quad (2)$$

where $g_1 = p_a(R_1)$ is the arithmetic genus of the curve R_1 , and $\iota_1 = 2n_{sp} + 2c_{sp} + c_{pp}$.

We apply the main inequality (2) to the proof of the Chisini conjecture in the case of generic pluricanonical coverings. Let S be a minimal model of a surface of general type. According to a theorem of Bombieri [BPV], if $m \geq 5$, then the m -canonical map $\varphi_m : S \rightarrow \mathbb{P}^{p_m-1}$, defined by the complete linear system numerically equivalent to $|mK_S|$, is a birational morphism, which blows down (-2) -curves on S . Then the canonical model $X = \varphi_m(S)$ has at most A-D-E-singularities. A generic projection $f : X \rightarrow \mathbb{P}^2$ is called a *generic m -canonical covering* for S . We prove the following theorem.

Theorem 0.3 *Let S_1 and S_2 be minimal models of surfaces of general type with the same (K_S^2) and $\chi(S)$, and let $f_1 : X_1 \rightarrow \mathbb{P}^2$, $f_2 : X_2 \rightarrow \mathbb{P}^2$ be generic m -canonical coverings with the same discriminant curve. Then for $m \geq 5$ the coverings f_1 and f_2 are equivalent.*

Consider a subvariety $\mathcal{H} \subset \text{Hilb} \times \text{Gr}$, parametrizing m -canonical coverings. Here Hilb is a subscheme of the Hilbert scheme, parametrizing numerically m -canonical embeddings $X \subset \mathbb{P}^M$ of surfaces with A-D-E-singularities and fixed (K_S^2) and $\chi(S)$, Gr is the Grassmann variety of projection centres from \mathbb{P}^M to \mathbb{P}^2 , and \mathcal{H} consists of pairs $(X \subset \mathbb{P}^M, L)$ such, that a restriction to X of a projection with centre L is a generic covering. By theorem 0.1 there is a one-to-one correspondence between the set of irreducible (respectively, connected) components of Hilb and \mathcal{H} . Let $h : \mathcal{H} \rightarrow \mathbb{P}^\nu$ be a map, taking a covering to its discriminant curve. Denote by \mathcal{D} a variety of plane curves of degree d with A-D-E-singularities, among which the number of nodes $\geq n_p$, and the number of cusps $\geq c_p$, where d, n_p, c_p are defined by invariants of S (see §6). By theorem 0.3 it follows (cf. [K], §5)

Corollary. The map, induced by h , from the set of irreducible (respectively, connected) components of the variety \mathcal{H} to the set of irreducible (respectively, connected) components of the variety \mathcal{D} is injective.

The proof of the main inequality (2) in [K] in the case of non-singular surfaces runs as follows. To compare two coverings f_1 and f_2 , a normalization X of the fibre product $X_1 \times_{\mathbb{P}^2} X_2$ is considered. Let $g_i : X \rightarrow X_i$, $i = 1, 2$, be the corresponding mappings to the factors. The preimage $g_1^{-1}(R_1) = R + C$ falls into two parts, where R is the curve mapped by g_2 to R_2 , and C is the curve mapped by g_2 to C_2 . If f_1 and f_2 are nonequivalent, then the surface X is irreducible, and if X_i are non-singular, then X is non-singular too. The main inequality is obtained by applying the Hodge index theorem to the pair of divisors R and C on X . We use the same idea also in the case of surfaces with A-D-E-singularities. For this we carry out the local analysis of the normalization of the fibre product X in the case of generic coverings of surfaces with A-D-E-singularities.

In §1 we generalize to the case of surfaces with A-D-E-singularities the theorem on generic projections. In §2 a local analysis of a normalization of the fibre product X is carried out. In §3 we investigate the canonical cycle of an A-D-E-singularity, with the help of which we compute numerical invariants of a generic covering in §4. In §5 the main inequality (2) is proved. Finally, in §6 the Chisini conjecture for generic m -canonical coverings of surfaces of general type is proved.

1 Singularities of a generic projection of a surface with A-D-E-singularities.

In this section we prove Theorem 0.1.

1.1. *A generic projection to \mathbb{P}^3 .* Let $X \subset \mathbb{P}^r$ be a surface of degree $\deg X = N$ with at most isolated hypersurface singularities x_1, \dots, x_k , i.e. such that the dimension of the tangent spaces $\dim T_{X, x_i} = 3$. Denote by $\pi_L : \mathbb{P}^r \setminus L \rightarrow \mathbb{P}^{e-1}$ a projection from a linear subspace L of codimension e . It can be obtained as a composition of projections with centers at points. The Theorem 0.1 on projections of X to the plane ($e = 3$) is one of a series of theorems on generic projections for different e , beginning with projections from points ($e = r$) and finishing by projections to the line ($e = 2$), i.e. Lefschetz pencils.

A classical result is that, if $r > 5$ ($= 2 \dim X + 1$), then the projection from a generic point gives an isomorphic embedding of X into \mathbb{P}^{r-1} . It follows that, if $e \geq 6$, then the projection from a generic subspace L gives an isomorphic embedding of X into \mathbb{P}^{e-1} . In particular, by a generic projection the surface X is embedded into \mathbb{P}^5 . When projecting to \mathbb{P}^4 , $e = 5$, there appears isolated singularities on $\pi_L(X)$, which is not difficult to describe. To prove Theorem 0.1 we are going to consider a generic projection of X into \mathbb{P}^3 , $e = 4$, and to take advantage of the following theorem.

Theorem 1.1 *If $X \subset \mathbb{P}^r$ is a surface with at most isolated hypersurface singularities x_i , then the restriction of a projection $\pi_L : \mathbb{P}^r \setminus L \rightarrow \mathbb{P}^3$ with the centre in a generic subspace $L \subset \mathbb{P}^r$ of codimension 4 gives a birational map of X onto a surface $Y \subset \mathbb{P}^3$, which is an isomorphism outside the double curve $D \subset X$ not passing through the points x_i , and Y has, except the points $\pi_L(x_i)$, at most ordinary singularities – the double curve $\Delta = \pi_L(D)$, on which there lie a finite number of ordinary triple points and a finite number of pinches. In neighbourhoods of these points in appropriate local analytic coordinates Y has normal forms as follows: $uv = 0$ for ordinary double points, $uvw = 0$ for ordinary triple points, $u^2 - vw^2 = 0$ for pinches (or "Whitney umbrellas").*

The contemporary proof of this theorem one can find in the textbook [G-H]. The presence of singular points x_i do not add extra troubles: we need only to see to the centre of the projection L not to intersect the tangent spaces T_{X, x_i} , $\dim T_{X, x_i} = 3$. A proof of this theorem one can find also in [M].

We want to prove that for a generic point $\xi \in \mathbb{P}^3$ the composition of projections π_L and $\pi_\xi : \mathbb{P}^3 \setminus \xi \rightarrow \mathbb{P}^2$, i.e. the projection $\mathbb{P}^r \setminus \pi_L^{-1}(\xi) \rightarrow \mathbb{P}^2$ with the centre $\pi_L^{-1}(\xi)$, restricted to X , $f = \pi_\xi \circ \pi_L|_X : X \rightarrow \mathbb{P}^2$, gives a covering satisfying the properties stated in Theorem 0.1.

1.2. *The disposition of lines with respect to a surface \mathbb{P}^3 .* To describe a projection π_ξ we need to investigate the disposition of lines $l \subset \mathbb{P}^3$ with respect to the surface Y . A line l is called *transversal* to Y at a point y , if it is transversal to the tangent cone to Y at this point. It means that $(l \cdot Y)_y = 1$, if $y \notin \text{Sing } Y$; $(l \cdot Y)_y = 2$, if $y \in \Delta \setminus \Delta_t$ and $(l \cdot Y)_y = 3$, if $y \in \Delta_t$. We denote by Δ_t and Δ_p the set of triple points and the set of pinches. If l is not transversal to Y at a point y , we say that it is tangent to Y at this point. A line l is called a *simple tangent* to Y at y , if $y \notin \text{Sing } Y$ and $(l \cdot Y)_y = 2$, or if $y \in \Delta \setminus (\Delta_t \cup \Delta_p)$ and $(l \cdot Y)_y = 3$, i.e. $(l \cdot Y)_y = 2$.

for one of two branches Y_i at the point y . A line l is called *stationary tangent*, respectively *simple stationary tangent* to Y at y , if $y \notin \text{Sing } Y$ and $(l \cdot Y)_y \geq 3$, respectively $= 3$. A line l is called *stationary tangent*, respectively *simple stationary tangent* to Y , if l is transversal to Y at all points, except one, at which l is stationary tangent, respectively simple stationary tangent, and, besides the other points of intersection $l \cap Y$ are non-singular on Y . Finally, l is called *simple bitangent*, if l is transversal to Y at all points, except two of them, at which the contact is simple, the tangent planes at them are distinct, and, besides, $l \cap \text{Sing } Y = \emptyset$. We want to prove that for a generic point $\xi \in \mathbb{P}^3$ all lines $l \ni \xi$ are at most simple bitangents and simple stationary tangents with respect to Y .

To study the disposition of lines $l \subset \mathbb{P}^3$ with respect to Y , we consider the Grassmann variety $G = G(1, 3)$ and the flag variety $\mathbb{F} = \{(\xi, l) \in \mathbb{P}^3 \times G \mid \xi \in l\}$. There are two projections $pr_1 : \mathbb{F} \rightarrow \mathbb{P}^3$ and $pr_2 : \mathbb{F} \rightarrow G$, which are \mathbb{P}^2 - and \mathbb{P}^1 -bundles respectively; $\dim \mathbb{F} = 5$, and $\dim G = 4$. In the sequel we consider points $\xi \in \mathbb{P}^3$ as centres of projection $\pi_\xi : \mathbb{P}^3 \setminus \xi \rightarrow \mathbb{P}^2$. The fibre $pr_1^{-1}(\xi) \simeq \mathbb{P}^2$ is mapped by the projection pr_2 isomorphically onto $\mathbb{P}_\xi^2 \subset G$. For $\xi \in \mathbb{P}^3$ there is a section $s_\xi : \mathbb{P}^3 \setminus \xi \rightarrow \mathbb{F}$ of the projection pr_1 , $y \mapsto (y, \overline{\xi y})$. Then π_ξ coincides with the restriction of the projection pr_2 to $s_\xi(\mathbb{P}^3 \setminus \xi)$.

Firstly, we consider the case, when a surface Y is non-singular, and then we describe the necessary modifications and supplements in the case, when there is a double curve Δ and isolated singularities s_i on Y .

Consider a filtration of the variety \mathbb{F} by subvarieties

$$Z_k = \{(\xi, l) \in \mathbb{F} \mid (l \cdot Y)_\xi \geq k\}.$$

Then $Z_1 = pr_1^{-1}(Y)$, $\dim Z_1 = 4$. Over a generic point $l \in G$ the map $\varphi = pr_{2|Z_1} : Z_1 \rightarrow G$ is an unramified covering of degree N . If there are no lines on Y , then φ is a finite covering, and Z_2 is the ramification divisor of the covering.

Now consider restrictions of the projection pr_1 . The variety Z_2 is isomorphic to a projectivized tangent bundle, $Z_2 \simeq \mathbb{P}(\Theta_Y)$, and $Z_2 \rightarrow Y$ is a \mathbb{P}^1 -fibre bundle, $\dim Z_2 = 3$. At a generic point $y \in Y$ there are two asymptotic directions l_1 and l_2 in $T_{Y,y}$, for which $(l_1 \cdot Y)_y$ and $(l_2 \cdot Y)_y \geq 3$. Therefore, over a generic point the restriction of pr_1 onto Z_3 , $\psi : Z_3 \rightarrow Y$, is a two-sheeted covering, the branch curve of which $P \subset Y$ is the parabolic curve consisting of points with coinciding asymptotic directions. Some fibres of the projection pr_1 are exceptional curves of the map ψ . Their images on Y are points y , at which the restriction of the second differential of the local equation of Y onto the tangent plane $T_{Y,y}$ vanishes. Such points y are called the *planar points* of the surface Y . The curve $H = \psi(Z_4) \subset Y$ consists of points y , at which at least one of the numbers $(l_i \cdot Y)_y \geq 4$ (H is a curve, if the surface Y is not a quadric).

1.3. Absence of non simple stationary tangents. Consider a product $Y \times \mathbb{F} \subset Y \times \mathbb{P}^3 \times G$ and projections pr'_1 and pr'_2 onto $Y \times \mathbb{P}^3$ and $Y \times G$. We can consider the varieties Z_k as subvarieties in $Y \times G \subset \mathbb{P}^3 \times G$. Consider a variety

$$I_4 = \{(y; \xi, l) \in Y \times \mathbb{F} \mid (l \cdot Y)_y \geq 4\} = (pr'_2)^{-1}(Z_4).$$

The projection $pr'_2 = id_Y \times pr_2$, as well as $pr_2 : \mathbb{F} \rightarrow G$, is a \mathbb{P}^1 -bundle. Therefore, $\dim I_4 = 2$ and $\dim \Sigma_4 \leq 2$, where $\Sigma_4 = p_2(I_4)$, and p_2 is a projection of $Y \times \mathbb{P}^3 \times G$ to \mathbb{P}^3 . Then, if $\xi \in \mathbb{P}^3 \setminus \Sigma_4$, we have that $(l \cdot Y)_y \leq 3$ for any line $l \ni \xi$ at any point $y \in Y$.

1.4. Absence of non simple bitangents. Consider a variety $\Sigma_{2,3} \subset \mathbb{P}^3$, made up of non simple bitangents, and show that $\Sigma_{2,3} \leq 2$. Consider a product $Y \times Y \times \mathbb{F} \subset Y \times Y \times \mathbb{P}^3 \times G$ and subvarieties $I_{i,j}$, which are closures of

$$I_{i,j}^0 = \{(y_1, y_2; \xi, l) \in Y \times Y \times \mathbb{F} : (Y \cdot l)_{y_1} \geq i, (Y \cdot l)_{y_2} \geq j, y_1 \neq y_2\}.$$

Denote a projection of $Y \times Y \times \mathbb{F}$ to $Y \times Y \times G$ by pr_2'' , and let $pr_2''(I_{i,j}) = \tilde{I}_{i,j}$. The projection pr_2'' and its restriction to $I_{i,j}$, $I_{i,j} \rightarrow \tilde{I}_{i,j}$ are \mathbb{P}^1 -bundles.

Lemma 1.1

$$\dim I_{2,3} \leq 2.$$

Proof. Consider subvarieties

$$Y \times Y \times G \supset \tilde{I}_{1,1} \supset \tilde{I}_{2,1} \supset \tilde{I}_{2,2} \supset \tilde{I}_{2,3},$$

and let q_1 be a projection onto the first factor. Obviously, $\tilde{I}_{1,1}$ is an irreducible variety of dimension $\dim \tilde{I}_{1,1} = 4$, birationally isomorphic to $Y \times Y$. The projection $q_1 : \tilde{I}_{2,1} \rightarrow Y$ is a fibration, fibers of which are curves $q_1^{-1}(y) \simeq C_y$, where

$$C_y = Y \cap T_{Y,y}.$$

The curve C_y has a singularity at the point y , which is a node for a generic point y .

Furthermore, the restriction of the projection to $\tilde{I}_{2,2}$, $q_1 : \tilde{I}_{2,2} \rightarrow Y$, is surjective, and its fibre over a point $y \in Y$ is

$$q_1^{-1}(y) = \{(y, y', l) \mid l \subset T_{Y,y} \text{ and } l \text{ is tangent to } Y \text{ at } y'\}.$$

We want to prove that $q_1(\tilde{I}_{2,3})$ doesn't coincide with Y , i.e. the embedding $Y \subset \mathbb{P}^3$ possesses the following property (L_1) : there exists a point $y \in Y$ such that all lines $l \subset T_{Y,y}$, passing through y , have at most simple contact with $C_y \setminus \{y\}$. We prove this below in 1.6 (Proposition 1.2) under the assumption that the embedding $Y \subset \mathbb{P}^3$ is obtained by a projection of an embedding "improved" by a Veronese embedding v_k , $k \geq 2$.

Thus, $\dim q_1(\tilde{I}_{2,3}) \leq 1$. A generic fibre of the map $q_1 : \tilde{I}_{2,3} \rightarrow Y$ is of dimension zero (it being one, Y is a ruled surface and we obtain a contradiction to the property (L_1)), therefore, $\dim \tilde{I}_{2,3} \leq 1$ and, consequently, $\dim I_{2,3} \leq 2$. ■

Set $\Sigma_{2,3} = p_3(I_{2,3})$, where p_3 is a projection of $Y \times Y \times \mathbb{P}^3 \times G$ to \mathbb{P}^3 . It follows from Lemma 1.1 that $\dim \Sigma_{2,3} \leq 2$. If $\xi \notin \Sigma_{2,3}$, then any line $l \ni \xi$, touching Y at two points y_1 and y_2 , has a simple contact at these points.

1.5. Absence of 3-tangents. Consider a product $Y \times Y \times Y \times \mathbb{F} \subset Y \times Y \times Y \times \mathbb{P}^3 \times G$ and subvarieties $I_{i,j,k}$, which are closures of

$$I_{i,j,k}^0 = \{(y_1, y_2, y_3; \xi, l) \in Y \times Y \times Y \times \mathbb{F} \mid (Y \cdot l)_{y_1} \geq i, (Y \cdot l)_{y_2} \geq j, (Y \cdot l)_{y_3} \geq k\},$$

where $y_1 \neq y_2 \neq y_3 \neq y_1$. Denote a projection of $Y \times Y \times Y \times \mathbb{F}$ onto $Y \times Y \times Y \times G$ by $pr_2^{(3)}$, and let $\tilde{I}_{i,j,k} = pr_2^{(3)}(I_{i,j,k})$. As above, it is clear that $\dim \tilde{I}_{1,1,1} = 4$, and $pr_1^{(3)}$ being a \mathbb{P}^1 -bundle, we have $\dim I_{1,1,1} = 5$.

Lemma 1.2

$$\dim I_{2,2,2} \leq 2.$$

Proof. Again consider a projection of $Y \times Y \times Y \times G$ and of its subvarieties

$$Y \times Y \times Y \times G \supset \tilde{I}_{1,1,1} \supset \tilde{I}_{2,1,1} \supset \tilde{I}_{2,2,1} \supset \tilde{I}_{2,2,2},$$

to the first factor. Consider $q_1 : \tilde{I}_{2,2,2} \rightarrow Y$. For a point $y \in Y$ we have $q_1^{-1}(y) = \{(y, y_2, y_3; l) \mid l \subset T_{Y,y}, l \text{ is tangent to } Y \text{ at points } y_2 \text{ and } y_3 \in l\}$. Just as in Lemma 1.1 it is sufficient to prove that $q_1(\tilde{I}_{2,2,2})$ doesn't coincide with Y . It means that there exists a point $y \in Y$, possessing the following property (L_2): none of the lines $l \subset T_{Y,y}$, passing through y , is not a bitangent, i.e. can't touch $C_y \setminus \{y\}$ at two different points. We prove this below in the following 1.6 (Proposition 1.2) under the assumption that the embedding $Y \subset \mathbb{P}^3$ is obtained by a projection of an embedding "improved" by a Veronese embedding v_k . ■

Set $\Sigma_{2,2,2} = p_4(I_{2,2,2})$, where p_4 is a projection of $Y \times Y \times Y \times \mathbb{P}^3 \times G$ to \mathbb{P}^3 . Then $\dim \Sigma_{2,2,2} \leq 2$ and if $\xi \notin \Sigma_{2,2,2}$, then any line $l \ni \xi$ touches Y at most at two points.

1.6. Embeddings with a Lefschetz property. The properties (L_1) and (L_2) in the two previous subsections mean that there exists a point $y \in Y$, for which the projection π_y of the curve $C_y \setminus \{y\} \subset T_{Y,y} \simeq \mathbb{P}^2$ from the point y is a Lefschetz pencil. Thus, to prove Lemmas 1.1 and 1.2 it is necessary to prove the existence of a point $y \in Y$ possessing the following "Lefschetz property" (L) with respect to the embedding into \mathbb{P}^3 . We formulate it for a surface X embedded into a projective space of any dimension.

Let $X \subset \mathbb{P}^r$ be an embedding into the projective space. We say, that a hyperplane $L_1 \subset \mathbb{P}^r$ possesses a property (L), if the curve $X \cap L_1$ has at most one node, i.e. L_1 touches X at a unique point x , at which the curve $X \cap L_1$ has an ordinary quadratic singularity. In other words, the point $[L_1] \in \check{\mathbb{P}}^r$, corresponding to L_1 , is a non-singular point of the dual variety X^\vee . We say that a pair (L_1, L_3) , where $L_3 \subset L_1$ is a linear subspace of dimension $r - 3$, possesses a property (L), if : L_1 possesses the property (L), $x \in L_3$, and a projection of the curve $X \cap L_1 \rightarrow \mathbb{P}^1$ from the centre L_3 is a Lefschetz pencil, i.e. any fibre of this (rational) mapping contains one singular point, and this point is at most quadratic (is of multiplicity 2). We say that an embedding $X \subset \mathbb{P}^r$ possesses a property (L), if $\exists x \in X$, for which $L_1 = T_{X,x}$ possesses the property (L), and L_1 can be added to a pair (L_1, L_3) with the property (L).

It is clear that, if a pair (L_1, L_3) possesses the property (L) and $Y \subset \mathbb{P}^3$ is obtained from X by projection from a centre $L_4 \subset L_3$, $\dim L_4 = r - 4$, then the embedding $Y \subset \mathbb{P}^3$ possesses the property (L).

Proposition 1.1 *If $S \subset \mathbb{P}^q$ is an embedding of a non-singular surface, and $X \subset \mathbb{P}^{rk}$ is its composition with the Veronese embedding v_k defined by polynomials of degree k , then the embedding $X \subset \mathbb{P}^{rk}$ possesses the property (L).*

Proof. Consider the hyperplane L_1 corresponding to a point $[L_1] \in X^\vee \setminus \text{Sing } X^\vee$. Then the curve $C = X \cap L_1$ contains a unique singular point – a node $x \in C$. Let $i : C \rightarrow X$ be the embedding. Consider a projection $\pi_{k,x} : \mathbb{P}^{rk} \setminus x \rightarrow \mathbb{P}^{rk-1}$ from the point x . To prove Proposition 1.1 it is enough to show that the image $\pi_{k,x}(C)$ is a non-singular curve in \mathbb{P}^{rk-1} . For then, if

L'_3 , $\dim L'_3 = r_k - 3$, is a centre of projection $\mathbb{P}^{r_k-1} \setminus L'_3 \rightarrow \mathbb{P}^1$, which is a Lefschetz pencil for $\pi_{k,x}(C)$, then, obviously, the pair (L_1, L_3) , where $L_3 = \pi_{k,x}^{-1}(L'_3) \cap L_1$, possesses the property (L).

Let I_x be the ideal sheaf of the point x on S , and $\mathcal{O}_S(1)$ be the sheaf of hyperplane sections. Under the identification $v_k : S \simeq X$, the map $\pi_{k,x}$ is given by sections of $H^0(S, \mathcal{O}_S(k) \otimes I_x)$. Let $k = 2$ and let $\sigma : \bar{S} \rightarrow S$ be a σ -process with centre at the point x . We can assume that \bar{S} is embedded into \mathbb{P}^{r_2-1} , where $r_2 - 1 = q(q+3)/2$, and the rational map $\sigma^{-1} : S \rightarrow \bar{S}$ is given by sections of $H^0(S, \mathcal{O}_S(2) \otimes I_x)$, i.e. it coincides with $\pi_{2,x}$. Since the proper transform $\bar{C} = \sigma^{-1}(C) \subset \bar{S}$ is a non-singular curve, we obtain Proposition 1.1 in the case $k = 2$. Besides, note that sections of $i^*(H^0(S, \mathcal{O}_S(2) \otimes I_x))$ give an embedding of \bar{C} into \mathbb{P}^{r_2-1} . Consequently, for $k > 2$ sections of $i^*(H^0(S, \mathcal{O}_S(k) \otimes I_x))$ also give an embedding of \bar{C} into \mathbb{P}^{r_k-1} , since there is a natural injection $H^0(S, \mathcal{O}_S(k-2)) \otimes H^0(S, \mathcal{O}_S(2) \otimes I_x) \subset H^0(S, \mathcal{O}_S(k) \otimes I_x)$, and sections of $H^0(S, \mathcal{O}_S(k-2))$ have no base points and fixed components. Therefore, sections of $i^*(H^0(S, \mathcal{O}_S(k) \otimes I_x))$ separate points and tangent directions on \bar{C} . ■

We say that a linear subspace L_4 of dimension $r-4$ possesses a property (L) with respect to an embedding $X \subset \mathbb{P}^r$, if the projection π_{L_4} to \mathbb{P}^3 from the centre L_4 maps X onto a surface $Y = \pi_{L_4}(X)$ with ordinary singularities.

As is known (see [G-H]), there is an open subset U in the Grassmannian $G_4 = Gr(r-4, r)$, points of which correspond to linear subspaces with the property (L).

Proposition 1.2 *If an embedding $X \subset \mathbb{P}^r$ possesses the property (L), then there exists a linear subspace L_4 with the property (L), which can be added to a flag $L_1 \supset L_3 \supset L_4$ such that the pair (L_1, L_3) possesses the property (L). In other words, there exists a projection to \mathbb{P}^3 , for which the embedding $Y \subset \mathbb{P}^3$, where Y is the image of X , possesses the property (L).*

Proof. Let $G_1 = \check{\mathbb{P}}^r$ be the dual space to \mathbb{P}^r , $G_3 = G(r-3, r)$ be the Grassmann variety of linear subspaces L_3 of dimension $r-3$, and $\mathbb{F} = \mathbb{F}_{1,3,4} \subset \check{\mathbb{P}}^r \times G_3 \times G_4$ be the variety of flags $L_1 \supset L_3 \supset L_4$. Let X^\vee be the dual variety, $W \subset X \times X^\vee \subset \mathbb{P}^r \times \check{\mathbb{P}}^r$ be a closed subvariety $W = \{(x, L_1) : L_1 \supset T_{X,x}\}$. Then the projection of $W \rightarrow X^\vee$ is an isomorphism over $X_0^\vee = X^\vee \setminus \text{Sing } X^\vee$, $W_0 \simeq X_0^\vee$.

Denote by $Z \subset X \times \mathbb{F}$ a closed subvariety

$$Z = \{(x; L_1 \supset L_3 \supset L_4) \mid (x, L_1) \in W, L_3 \ni x\},$$

and by $Z_0 \subset Z$ an open subset: $(x, L_1) \in W_0$. Then Z is an irreducible variety. Consider a projection $Z_0 \rightarrow W_0$. The fibres are not empty by the previous proposition, and each of the fibres contains an open set of points z , for which the pair (L_1, L_3) possesses the property (L) (because the centres of projections for Lefschetz pencils form an open set). Therefore, Z contains an open set Z_L , for points of which the pair (L_1, L_3) possesses the Lefschetz property.

Obviously, the map $Z \rightarrow G_4$ is surjective. Therefore, $p_4^{-1}(U)$, where p_4 is a projection of Z to G_4 , is a non empty Zariski open set. Then $Z_L \cap p_4^{-1}(U)$ is not empty, and if $(x; L_1 \supset L_3 \supset L_4)$ is a point of this set, then L_4 possesses the desired property. ■

1.7. Projecting in a neighbourhood of a generic point of the double curve Δ . Now let $Y \subset \mathbb{P}^3$ has ordinary singularities along the double curve Δ and isolated singularities $s_i \in$

$Y \setminus \Delta, i = 1, \dots, k$, which are double planes. Under the incidence varieties, defined in the previous subsections, we mean the closures of the corresponding varieties, initially defined for an open surface $Y \setminus \text{Sing } Y$.

Consider $Y \times \mathbb{F}$. In addition to notations in 1.3, let q_1 and q_2 the projections of $Y \times G$ to Y and G . Consider the intersection $\tilde{A} = (\Delta \times G) \cap Z_3$. Then the restriction of the projection $Y \times G \rightarrow Y$ to \tilde{A} , $\tilde{A} \rightarrow \Delta$, is a covering of degree 4 over a generic point : at a point $y \in \Delta$ there are two asymptotic directions for each of two branches of Y at y . Therefore, \tilde{A} is a curve. Set $A = (pr'_2)^{-1}(\tilde{A})$. It is a ruled surface. Set $\Sigma_\Delta = p_2(A)$. Then, if $\xi \notin \Sigma_\Delta$, we have that for a generic point $y \in \Delta$ the lines $l \ni \xi$ have at most simple contact with branches of Y .

Denote by Σ_0 the union of planes in \mathbb{P}^3 composing the tangent cones at the rest points of Δ , including triple points and pinches, and also at singular points $s_i \in Y$.

1.8. Projecting in a neighbourhood of a triple point. If $\xi \notin \Sigma_0$, then in a neighbourhood of a point $y \in \Sigma_t$ all lines $l \ni \xi$ are transversal to each of the three branches of Y at y , and therefore, locally these branches are mapped isomorphically onto \mathbb{P}^2 .

1.9. Projecting in a neighbourhood of a pinch. In a neighbourhood of a pinch $y \in Y$ there are coordinates, by which Y is locally defined by an equation $u^2 = vw^2$. The double curve $\Delta \subset Y$ is a line $u = w = 0$, and the tangent cone $C_{Y,y}$ to Y at y has an equation $u = 0$. In a neighbourhood of a pinch a normalization $\mathbb{C}^2 \rightarrow Y$ is defined by formulae $u = tw, v = t^2, w = w$. Since X is non-singular and π_L is a finite map, we can assume that the projection π_L is the normalization. If a point ξ does not belong to the tangent cone $C_{Y,y}$, then the projection π_ξ locally is a map of gedrr 2. A projection $f : X \rightarrow \mathbb{P}^2$ a neighbourhood of the preimage of a pinch is a two-sheeted covering of non-singular varieties, and, hence, locally is defined by equations $v = t^2, w = w$. Thus, the curve $\bar{R} \subset Y$ goes through the pinch, and pinches are projected to non-singular points of the discriminant curve B .

1.10. Normal forms of a generic projection at points of the ramification curve.

Lemma 1.3 ([A]) *Let $(X, 0) \subset (\mathbb{C}^3, 0)$ be a non-singular surface, and $(\mathbb{C}^3, 0) \rightarrow (\mathbb{C}^2, 0)$ be a smooth morphism, the restriction of which $f : X \rightarrow \mathbb{C}^2$ is a finite covering of degree μ . Then one can choose local coordinates x, y in \mathbb{C}^2 and x, y, z in \mathbb{C}^3 such, that X is defined by an equation*

$$y = z^\mu + \lambda_1(x)z^{\mu-2} + \dots + \lambda_{\mu-2}(x)z,$$

and f is a projection along z axis.

Proof. This is Lemma 1 in Arnol'd paper [A]. It is obtained, if we consider the covering f as a 2-parameter family of 0-dimensional hypersurface singularities of multiplicity μ , and, consequently, f is induced by the miniversal deformation of the singularity of type $A_{\mu-1}$. ■

We proved that at a generic point P of the ramification curve a projection $f : X \rightarrow \mathbb{P}^2$ is of degree $\mu = 2$, and at isolated points it is of degree $\mu = 3$. By Lemma 1.3 for $\mu = 2$ we obtain that at a generic point of the ramification curve a generic projection is equivalent to a projection of the surface $X : x = z^2$ to the x, y -plane, i.e. it is a fold. For $\mu = 3$ we obtain

Corollary 1.1 *For a non-singular surface X a finite covering $X \rightarrow \mathbb{C}^2$ of degree 3 locally is a projection to the x, y -plane of one of the surfaces*

$$y = z^3 + x^k z, \quad k = 1, 2, \dots, \quad \text{or} \quad y = z^3 \quad (k = \infty).$$

In the case $k \neq \infty$ the ramification curve C is reduced and has an equation $3z^2 + x^k = 0$ in local coordinates x, z on X . The curve C is non-singular only for $k = 1$. The discriminant curve B has an equation $4x^{3k} + 27y^2 = 0$, i.e. B is a cusp. It is an ordinary cusp only for $k = 1$.

Proof. By lemma 1.3 we have that X is defined by an equation $y = z^3 + \lambda_1(x)z$. We obtain the stated normal form of the covering f , where k is the order of vanishing of $\lambda_1(x)$ at the point $x = 0$. The ramification curve C is defined by equation $J = 0$, where $J = 3z^2 + x^k$ is the Jacobian of the covering f . The discriminant curve B is defined by 0th Fitting ideal $F_0(f_*\mathcal{O}_C)$ of the sheaf $f_*\mathcal{O}_C$. To obtain an equation of B — the generator of the Fitting ideal, we need to take a finite presentation $f_*\mathcal{O}_X \xrightarrow{J} f_*\mathcal{O}_X \rightarrow f_*\mathcal{O}_C \rightarrow 0$ of the sheaf $f_*\mathcal{O}_C$, where $(f_*\mathcal{O}_X)_0 = \mathbb{C}\{x, z\} = \mathbb{C}\{x, y\} \cdot 1 \oplus \mathbb{C}\{x, y\}z \oplus \mathbb{C}\{x, y\}z^2$, and to compute a determinant of the $\mathbb{C}\{x, y\}$ -linear map J , which is a multiplication by the Jacobian J . ■

Now we show that for a generic projection the discriminant curve B has at most ordinary nodes and cusps. Let $b \in B$ be a point corresponding to a bitangent l under projecting $\pi_\xi : \mathbb{P}^3 \setminus \xi \rightarrow \mathbb{P}^2$ from a point ξ . Let l touches Y at points P_1 and P_2 , to which correspond branches B_1 and B_2 of the discriminant curve B at a point b . We have to show that for a generic projection the point b is a node, i.e. the branches B_1 and B_2 have different tangents. Determine where does the centres ξ of "bad" projections lie. Let a line $\lambda \subset \mathbb{P}^2$ is a common tangent to branches B_1 and B_2 at a point b . Then the plane $\pi_\xi^{-1}(\lambda)$ is bitangent — it touches the surface Y at points P_1 and P_2 . Consider the dual surface $Y^\vee \subset \check{\mathbb{P}}^3$. Then the point $[\pi_\xi^{-1}(\lambda)] \in \text{Sing } Y^\vee = \gamma^\vee$. Set $\gamma = \tau^{-1}(\gamma^\vee)$, where $\tau : Y \rightarrow Y^\vee$ is the Gauss map. Let $\Sigma_u \subset \mathbb{P}^3$ be a ruled surface composed by lines P_1P_2 , where $P_1, P_2 \in \gamma$, $\tau(P_1) = \tau(P_2) = [\pi_\xi^{-1}(\lambda)]$. Then, if $\xi \notin \Sigma_u$, then at points b , corresponding to bitangents l , the curve B has at most nodes.

Now let $b \in B$ be a point corresponding to a stationary tangent l at a point $P \in Y$. As was noted above, in a neighbourhood of P the projection π_ξ is equivalent to a projection of a surface $y = z^3 + x^k z$ to the x, y -plane. We have to show that for a generic projection the exponent $k = 1$. The fact is that, if $k > 1$, then the point P is a planar point of the surface Y . Excepting the centres of projection lying in tangent planes to Y at planar points, we obtain that in a neighbourhood of a point with $\mu = 3$ the projection f is equivalent to a projection of a surface $y = z^3 + xz$ to the x, y -plane, i.e. it is a pleat.

1.11. Projecting in a neighbourhood of an isolated double plane singularity.

Lemma 1.4 *If $(X, 0) \subset (\mathbb{C}^3, 0)$ is an (isolated) double plane singularity $z^2 = h(x, y)$, $\pi : X \rightarrow \mathbb{C}^2$ be a projection from any point $p \in \mathbb{C}^3$, not lying in the tangent cone $z = 0$, then the ramification curve of π is reduced, and the discriminant curve $B \subset \mathbb{C}^2$ is locally analytically isomorphic to the singularity $h(x, y) = 0$.*

Proof. The singularity $(X, 0)$ is of multiplicity 2. Therefore, π is a covering of degree 2, and, consequently, is locally a projection of a double plane $w^2 = g(u, v)$ to the (u, v) -plane.

Thus, the germs of singularities $h(x, y)$ and $g(u, v)$ are stably isomorphic, and hence isomorphic ([AGV] , vol.1). ■

2 Local structure of fibre products.

2.1. Local structure of a generic covering. Let $f : X \rightarrow \mathbb{P}^2$ be a generic covering of the plane by a surface X with A-D-E-singularities, and let $B \subset \mathbb{P}^2$ be the discriminant curve, $f^*(B) = 2R + C$. Singular points $o \in \text{Sing } X$ will be called *s-points* of the surface X (from the word singularity). In a neighbourhood of a s-point o the covering f is isomorphic to the projection to x, y -plane of a surface $z^2 = h(x, y)$, where $h(x, y)$ has one of the A-D-E-singularities. Singular points o on X correspond to singular points of the same type as o on B . With respect to f non-singular points of X are partitioned into *r-points* (from the word regular), at which the morphism f is étale, and *p-points* (from the words singularity of projection) – they are points of the ramification curve R . A p-point is either a fold (or a singular p-point of type A_1), in a neighbourhood of which $f : (x, z) \mapsto (x, y)$, $y = z^2$, or a pleat $o \in R \cap C$ (or a singular p-point of type A_2), in a neighbourhood of which $f : (x, z) \mapsto (x, y)$, $y = z^3 - 3xz$ (more details about this see in section 2.4 below).

The singular points of B ‘originated’ from singular points $\text{Sing } X$ we call *s-points*. There are additional singular points of type A_1 (nodes) and of type A_2 (cusps), which we call *p-nodes* and *p-cusps*. Over a generic point $b \in B$ there lie one fold and $N - 2$ r-points; over p-node there lie two folds and $N - 4$ r-points; over a p-cusp there lie one pleat and $N - 3$ r-points; over a s-node or a s-cusp, as over any s-point $b \in B$, there lie one singular point of X and $N - 2$ r-points.

2.2. Types of points on the fibre product. With respect to a pair of generic coverings $f_1 : X_1 \rightarrow \mathbb{P}^2$ and $f_2 : X_2 \rightarrow \mathbb{P}^2$ with the same discriminant curve $B \subset \mathbb{P}^2$ nodes and cusps on B are partitioned by this time into 4 types: *ss-*, *sp-*, *ps-* and *pp-nodes* and *cusps*. For example, a ps-node it is a node $b \in B$, such that there are two folds on X_1 over b , and on X_2 over b there is a singular point of type A_1 . The analogous terminology is used for the classification of points $\bar{x} = (x_1, x_2)$ on the fibre product $X^\times = X_1 \times_{\mathbb{P}^2} X_2$: we say about rs-points, ss-points, sp-points, etc. For example, we say that \bar{x} is a ps-point of type A_2 , if $x_1 \in X_1$ is a p-point of type A_2 , and $x_2 \in \text{Sing } X_2$ is a singular s-point of type A_2 .

In this section we describe the structure of a normalization $\nu : X = (X^\times)^{(\nu)} \rightarrow X^\times$ of the fibre product X^\times . Denote by g_1, g_2 and f the morphisms of X to X_1, X_2 and \mathbb{P}^2 . Since the normalization is defined locally, we can replace \mathbb{P}^2 by a neighbourhood of the point $0 \in \mathbb{C}^2$ and to assume that X_1 and X_2 are neighbourhoods of points $x_1 \in X_1$ and $x_2 \in X_2$. We pass on to an item-by-item examination of all possible types of points $\bar{x} = (x_1, x_2) \in X^\times$. We do it up to the permutation of factors X_1 and X_2 .

At first we consider quite trivial cases.

2.2.1. If \bar{x} is a r^* -point (where $*$ = r, s, p), then X^\times at the point \bar{x} is locally the same as X_2 at the point x_2 , and $f^\times : X^\times \rightarrow \mathbb{C}^2$ is locally the same as $f_2 : X_2 \rightarrow \mathbb{C}^2$.

2.2.2. If \bar{x} is a 2×2 -point, i.e. x_1 and x_2 are points of ‘double planes’, $z_1^2 = h(x, y)$, $z_2^2 =$

$h(x, y)$, then $X^\times = X_1 \times_{\mathbb{C}^2} X_2$ in a neighbourhood of the point $\bar{x} = (x_1, x_2)$ is a surface in $\mathbb{C}^4 \ni (x, y, z_1, z_2)$, defined by equations $z_1^2 = h(x, y)$, $z_2^2 = h(x, y)$. We obtain that $z_1^2 = z_2^2$ and hence $X^\times = X_1^\times \cup X_2^\times$, where $X_1^\times : z_1^2 = h(x, y), z_2 = z_1$, $X_2^\times : z_2^2 = h(x, y), z_1 = -z_2$. The surfaces X_1^\times and X_2^\times meet along a curve $z_1 = z_2 = 0, h(x, y) = 0$. We obtain that a normalization $X = \widehat{X}^\times = X_1^\times \sqcup X_2^\times$ locally consists of two disjoint components X_1^\times and X_2^\times isomorphic to X_1 and X_2 .

In particular, we obtain a description of the normalization in a neighbourhood of a pp-point $(x_1, x_2) \in X^\times$ lying over a non-singular point of B , $B : x = 0$,

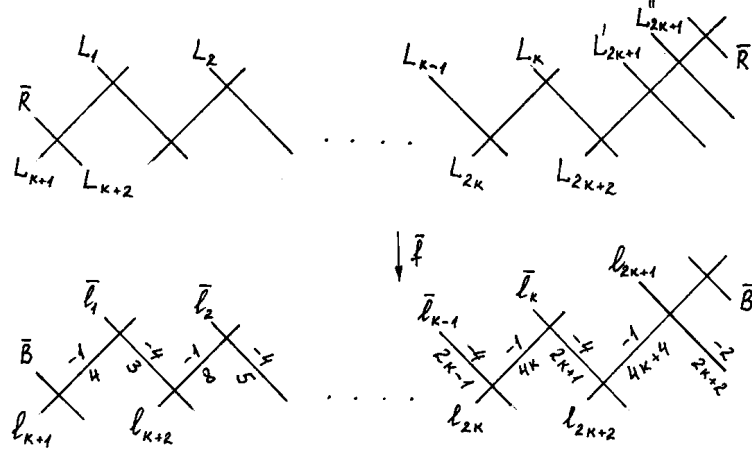


Fig. 1

$$g_1^*(R_1) = R' + R'', \quad g_2^*(R_2) = R' + R''.$$

Every ss-point is a 2×2 -point. Thus, in a neighbourhood of a ss-point the normalization has the same local structure as in the case of a pp-point above: X locally consists of two disjoint components isomorphic to X_1 and X_2 .

It remains to examine less trivial cases when \bar{x} is a pp- or sp-point of type A_1 or A_2 . This is done in the following two subsections.

2.3. On fibre product of double planes.

2.3.1. The ordinary quadratic singularity – the singularity of type A_1 on a surface $X_0 : z^2 = xy$ can be considered as a 2-sheeted covering of the plane $f_0 : X_0 \rightarrow \mathbb{C}^2$ branched along a node $B : xy = 0$. As is known, the singularity X_0 itself can be considered as a quotient singularity under the action of cyclic group $\mathbb{Z}_2 = \{\pm 1\}$, $X_0 = X/\mathbb{Z}_2$, where $X = \mathbb{C}^2 \ni (z_1, z_2)$, and a generator of \mathbb{Z}_2 acts by the rule: $z_1 \mapsto -z_1, z_2 \mapsto -z_2$. The factorization morphism $g_0 : X \rightarrow X_0$ is defined by formulae

$$x = z_1^2, \quad y = z_2^2, \quad z = z_1 z_2.$$

We obtain a 4-sheeted covering $f = f_0 \circ g_0 : X \rightarrow \mathbb{C}^2$, which can be considered as a factorization under the action of the group $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ on X . Then the factorization g_0 corresponds

to a subgroup of order two $\mathbb{Z}_2 = G_0 = \{(1, 1), (-1, -1)\}$, imbedded diagonally into G . In G there are two more subgroups of order two: $G_1 = \{1\} \times \mathbb{Z}_2$ and $G_2 = \mathbb{Z}_2 \times \{1\}$. Considering $X_1 = \mathbb{C}_2/G_1 \simeq \mathbb{C}_2$ and $X_2 = \mathbb{C}_2/G_2 \simeq \mathbb{C}_2$, we obtain two more decompositions of f and a commutative diagram

$$\begin{array}{ccccc}
 & & X = \mathbb{C}^2 \ni (z_1, z_2) & & \\
 & g_1 \swarrow & \downarrow g_0 & \searrow g_2 & \\
 (z_1, y) \in \mathbb{C}^2 = X_1 & & X_0 & & X_2 = \mathbb{C}^2 \ni (x, z_2) \ , \\
 & f_1 \searrow & \downarrow f_0 & \swarrow f_2 & \\
 & & (x, y) \in \mathbb{C}^2 & &
 \end{array} \quad (*)_2$$

where $g_1 : y = z_2^2$, $f_1 : x = z_1^2$, $g_2 : x = z_1^2$, $f_2 : y = z_2^2$.

Denote by $B_1 : x = 0$, $B_2 : y = 0$ the branches of $B : xy = 0$, and by $R' : z_1 = 0$, $R'' : z_2 = 0$ the branches of their proper transform $z_1 z_2 = 0$ on X .

The diagram $(*)_2$ shows that we can consider X as a normalization in three cases:

2.3.2. X is a normalization in a neighbourhood of a ps-point of type A_1 , $\bar{x} \in X_1^\times = X_1 \times_{\mathbb{C}^2} X_0$,

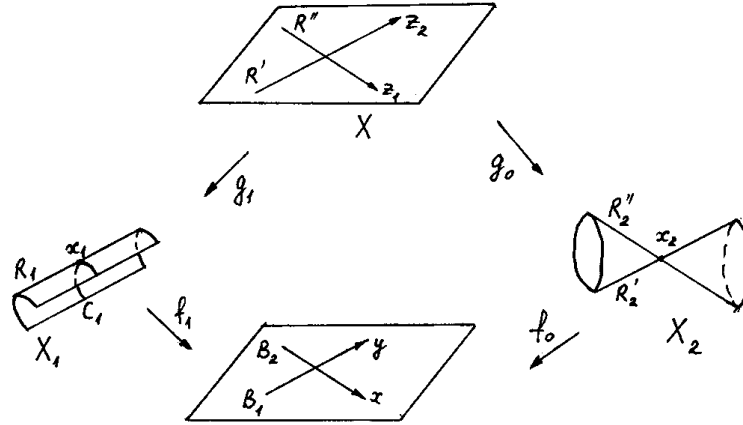


Fig. 2

$$f_1^*(B_1) = 2R_1, f_1^*(B_2) = C_1; g_1^*(R_1) = R', g_1^*(C_1) = 2R'';$$

$$f_0^*(B) = 2R_2 = 2(R_2' + R_2''); g_0^*(R_2') = R'; g_0^*(R_2'') = R''.$$

(g_0 is unramified outside the point $0 \in X_0$).

2.3.3. X is a normalization in a neighbourhood of a sp-point of type A_1 , $\bar{x} \in X_2^\times = X_0 \times_{\mathbb{C}^2} X_2$ (the case symmetric to 2.3.2.)

2.3.4. X is a normalization in a neighbourhood of a pp-point of A_1 , $\bar{x} \in X^\times = X_1 \times_{\mathbb{C}^2} X_2$, which is not a 2×2 -point, $f_1 : x = z_1^2$, $f_2 : y = z_2^2$.

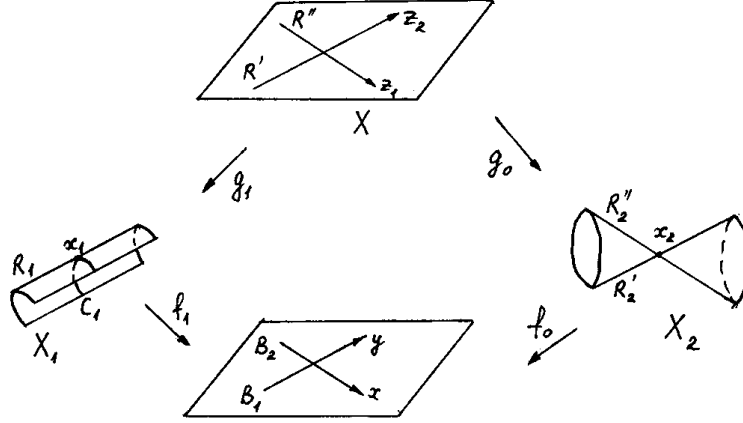


Fig. 3

$$f_1^*(B) = 2R_1 + C_1, \quad g_1^*(R_1) = R', \quad g_1^*(C_1) = 2R'',$$

$$f_2^*(B) = 2R_2 + C_2, \quad g_2^*(R_2) = R'', \quad g_2^*(C_2) = 2R'.$$

Using 2.3.2-2.3.4, now we can describe a normalization X over a neighbourhood of a node $b \in B$.

2.3.5 Over a neighbourhood of a ps-node $b \in B$ (as well as a sp-node) a normalization of X^\times in a neighbourhood of a ps-point looks like as

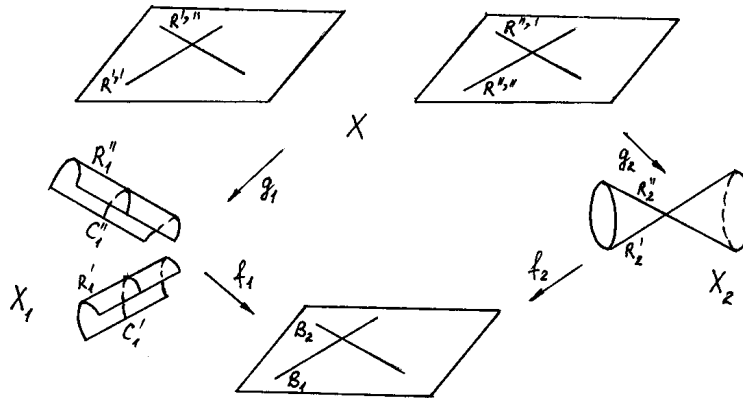


Fig. 4

$$g_1^*(R'_1) = R'^{'}, \quad g_1^*(C'_1) = 2R'^{'}, \quad g_1^*(R''_1) = R''^{'}, \quad g_1^*(C''_1) = 2R''^{'},$$

$$g_2^*(R'_2) = R'^{'}, \quad g_2^*(R''_2) = R'^{'}, \quad g_2^*(C'_2) = R'^{'}, \quad g_2^*(C''_2) = R''^{'},$$

On Fig. 4 the normalization in neighbourhoods of pr-, rs- and rr-points of X^\times is not pictured.

2.3.6. Over a neighbourhood of a pp-node $b \in B$ a normalization of X^\times in a neighbourhood of a pp-point looks like as:

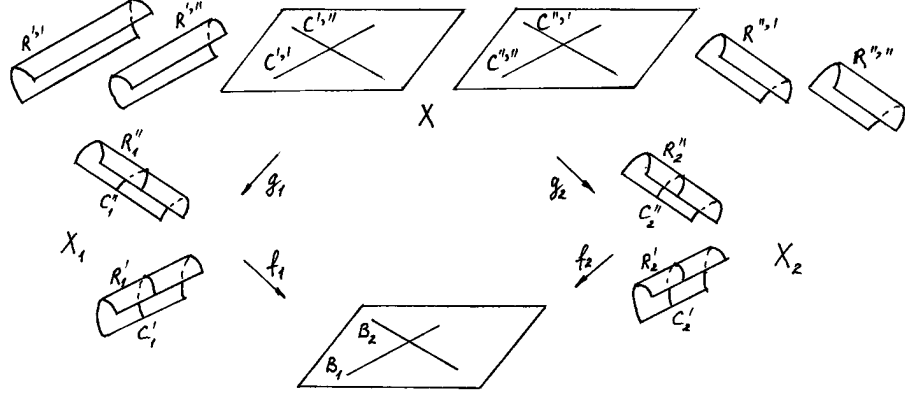


Fig. 5

$$g_1^*(R'_1) = R'^{'}, \quad g_1^*(R''_1) = R''^{'}, \quad g_1^*(C'_1) = R'^{'}, \quad g_1^*(C''_1) = R''^{'},$$

$$g_2^*(R'_2) = R'_2, \quad g_2^*(R''_2) = R'_2, \quad g_2^*(C'_2) = R'_2, \quad g_2^*(C''_2) = R'_2,$$

$$g_2^*(C'_3) = C'_3, \quad g_2^*(C''_3) = C'_3, \quad (g_1(C'_3) = C'_1, \quad g_1(C''_3) = C'_1).$$

2.4. On coverings of \mathbb{C}^2 unbranched outside a cusp $B : y^2 = x^3$. To describe a normalization of the fibre product in a neighbourhood of a sp- and pp-point of type A_2 in a natural context, we begin with reminding of a small topic from singularity theory.

2.4.1 The singularity of cuspidal type of a map (pleat) and the miniversal deformation of a singularity of type A_2 . A cusp $(B, 0) \subset (\mathbb{C}^2, 0)$ is defined by a germ of function $x^3 - y^2$ stable equivalent to a germ of function x^3 . It is a simple singularity of type A_2 . It is interesting that a cusp (a singularity of type A_2) appears also on the discriminant in the base of the miniversal deformation of the same singularity of type A_2 .

As is known, the miniversal unfolding of the function $t = z^3$ is

$$\mathbb{C} \times \mathbb{C}^2 \rightarrow \mathbb{C} \times \mathbb{C}^2, \quad (z, a_2, a_3) \mapsto (z^3 + a_2z + a_3, a_2, a_3).$$

The restriction of this map over $\{0\} \times \mathbb{C}^2$ gives a miniversal deformation F of a zero-dimensional singularity $z^3 = 0$, $\mathbb{C}^3 \supset X \xrightarrow{F} \mathbb{C}^2$. Here X is a surface $z^3 + a_2z + a_3 = 0$, and F is induced by projection onto (a_2, a_3) . The surface X is a graph of function $-a_3 = z^3 + a_2z$; z and a_2 are local coordinates on X ,

$$\begin{array}{ccc}
(a_2, z) \in \mathbb{C}^2 & \xrightarrow{\sim} & X \subset \mathbb{C}^3 \\
\searrow G & & \downarrow F \\
& & (a_2, a_3) \in \mathbb{C}^2
\end{array}
, \quad G : \begin{cases} a_2 &= a_2 \\ -a_3 &= z^3 + a_2 z. \end{cases}$$

We obtain a 3-sheeted covering $G : \mathbb{C}^2 \rightarrow \mathbb{C}^2$, the ramification curve of which R is defined by the equation $3z^2 + a_2 = 0$, and the discriminant (branch) curve $B = G(R)$ is defined by equation

$$4a_2^3 + 27a_3^2 = 0.$$

To bring the equation of B to the form $y^2 = x^3$, we make a substitution

$$a_2 = -3x, \quad a_3 = 2y,$$

and denote $\mathbb{C}^2 \simeq X$ by X_3 , and G by f_3 .

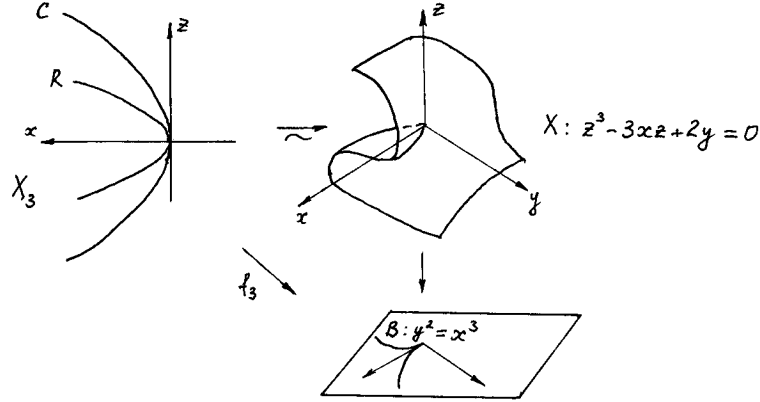


Fig. 6

We obtain a 3-sheeted covering $f_3 : X_3 \rightarrow \mathbb{C}^2$,

$$f_3 : x = x, \quad y = -\frac{1}{2}(z^3 - 3xz).$$

Then $x^3 - y^2 = x^3 - \frac{1}{4}(z^3 - 3xz)^2 = (x - z^2)^2(x - \frac{1}{4}z^2)$ and, consequently,

$$f_3^*(B) = 2R + C,$$

where $R : x = z^2$ is the ramification curve, and $C : x = \frac{1}{4}z^2$. Note that C and R are tangent of order two, $(C \cdot R) = 2$.

By Lemma 1.3 the singular point of the covering f_3 is uniquely characterized as a singular point of a 3-sheeted covering $f : X \rightarrow \mathbb{C}^2$ by a non-singular surface X , the discriminant curve of which is an ordinary cusp.

2.4.2 *The Viète map f_6 .* We produce a well known regular covering of \mathbb{C}^2 with group S_3 branched along a cusp $B : y^2 = x^3$, which appears to be a normalization of the fibre product in a neighbourhood of a sp-point of type A_2 . This covering naturally appears in singularity theory.

Consider a quotient of the space \mathbb{C}^3 under the action of permutation group S_3 . We get the Viète map

$$v : \mathbb{C}^3 \rightarrow \mathbb{C}^3, (z_1, z_2, z_3) \mapsto (a_1, a_2, a_3),$$

where $(z - z_1)(z - z_2)(z - z_3) = z^3 + a_1 z^2 + a_2 z + a_3$, i.e.

$$a_1 = -(z_1 + z_2 + z_3), a_2 = z_1 z_2 + z_2 z_3 + z_3 z_1, a_3 = -z_1 z_2 z_3.$$

The map v is a map of degree 6 unramified outside $\Delta = \cup_{i \neq j} \{z_i = z_j\}$, and $v(\Delta) = D$ is defined by the discriminant of a polynomial of degree three.

The action of S_3 on \mathbb{C}^3 is reducible: \mathbb{C}^3 is a direct sum $\mathbb{C}^3 = \mathbb{C} \oplus \mathbb{C}^2$ of invariant subspaces – of the line $\mathbb{C} = \{z_1 = z_2 = z_3\}$ and of the plane $\mathbb{C}^2 = \{z_1 + z_2 + z_3 = 0\}$. Consider the restriction of v to this plane \mathbb{C}^2 ,

$$(z_1, z_2, z_3) \in \mathbb{C}^3 \supset \{z_1 + z_2 + z_3 = 0\} = \mathbb{C}^2 \longrightarrow \mathbb{C}^2 = \{a_1 = 0\} \subset \mathbb{C}^3 \ni (a_1, a_2, a_3).$$

Set $\mathbb{C}^2 \cap \Delta = L$, $\mathbb{C}^2 \cap D = B$. Then L consists of three lines

$$L = L_1 + L_2 + L_3, \text{ where } L_i : z_j = z_k, z_1 + z_2 + z_3 = 0, \{i, j, k\} = \{1, 2, 3\},$$

and the curve $B : 4a_2^3 + 27a_3^2 = 0$ is defined by the discriminant of the polynomial $z^3 + a_2 z + a_3$. Since $\pi_1(\mathbb{C}^2 \setminus L) = \pi_1(\mathbb{C}^3 \setminus \Delta)$, $\pi_1(\mathbb{C}^2 \setminus B) = \pi_1(\mathbb{C}^3 \setminus D) = Br_3$, we obtain a covering $v : \mathbb{C}^2 \rightarrow \mathbb{C}^2$ of degree 6 unbranched apart from B . Denote this map by f_6 . In coordinates x, y , where $a_2 = -3x$, $a_3 = 2y$, this map

$$\mathbb{C}^2 = \{z_1 + z_2 + z_3 = 0\} = X_6 \xrightarrow{f_6} \mathbb{C}^2 \ni (x, y)$$

is defined by formulae

$$f_6 : x = -\frac{1}{3}(z_1 z_2 + z_2 z_3 + z_3 z_1), y = -\frac{1}{2} z_1 z_2 z_3,$$

the discriminant B has equation $y^2 = x^3$, and $f^*(B) = 2L = 2L_1 + 2L_2 + 2L_3$ (it is easy to see that $x^3 - y^2 = \frac{1}{4 \cdot 27}(z_2 - z_1)^2(z_3 - z_2)^2(z_1 - z_3)^2$ under condition $z_1 + z_2 + z_3 = 0$).

Consider a two-sheeted covering unbranched outside B

$$(x, y, w) \in \mathbb{C}^3 \supset X_2 \xrightarrow{f_2} \mathbb{C}^2 \ni (x, y),$$

where X_2 is defined by equation $w^2 = x^3 - y^2$, and f_2 is induced by projection. Such a structure has a generic covering $f : X \rightarrow \mathbb{P}^2$ in a neighbourhood of a s-point of type A_2 .

Lemma 2.1 ([C]) *If $f : (X, 0) \rightarrow (\mathbb{C}^2, 0)$ is a finite covering by a normal irreducible surface X , unbranched outside an ordinary cusp $B \subset \mathbb{C}^2$, and the ramification curve of which is reduced, i.e. $f^*(B) = 2R + C$ (R and C reduced curves), then f is equivalent to one of the coverings f_2, f_3 and f_6 . ■*

The proof is obtained by means of studying the possible monodromy homomorphisms $\rho : \pi_1 \rightarrow S_N$, where $\pi_1 = \pi_1(\mathbb{C}^2 \setminus B) = Br$ is the fundamental group of a cusp, and $N = \deg f$.

We obtain one more characterization of the covering f_3 as a finite covering $f : (X, 0) \rightarrow (\mathbb{C}^2, 0)$ by a normal irreducible surface, unbranched outside a cusp B , and with a reduced and non-singular ramification curve R .

2.4.3 *Description of a normalization of the fibre product in a neighbourhood of a sp-point of type A_2 .* The map f_6 factors through the maps f_2 and f_3 , and we have a commutative diagram

$$\begin{array}{ccccc}
 & & X_6 = \{z_1 + z_2 + z_3 = 0\} & & \\
 & g_3 \swarrow & \downarrow & \searrow g_2 & \\
 \{w^2 = x^3 - y^2\} = X_2 & & f_6 & & X_3 = \{z^3 - 3xz^2 + 2y = 0\}, \\
 & f_2 \searrow & \downarrow & \swarrow f_3 & \\
 & & (x, y) \in \mathbb{C}^2 \supset B : y^2 = x^3 & &
 \end{array} \tag{*_3}$$

where g_2 and g_3 are defined by formulae: x and y are defined by the same formulae as f_6 , and $z = z_1$ for g_2 , and $w = \frac{1}{6\sqrt{3}}(z_2 - z_1)(z_3 - z_2)(z_1 - z_3)$ for g_3 . It is easy to see that g_3 is a factorization under the action of a cyclic group $\mathbb{Z}_3 = \mathcal{A}_3 \subset S_3$, $X_2 = X_6/\mathcal{A}_3$, and g_2 is a factorization under the action of a cyclic group of order two $\mathbb{Z}_2 \simeq S_2 = \{(1), (2, 3)\} \subset S_3$.

By the property of universality of fibre products we have a morphism $X_6 \rightarrow X_2 \times_{\mathbb{C}^2} X_3$. The fibre product $X_2 \times_{\mathbb{C}^2} X_3$ is irreducible, since each its component Z is mapped onto X_2 and X_3 , and, therefore, the degree of $Z \rightarrow \mathbb{C}^2$ have to be divided by 2 and 3, i.e. have to be equal to 6. Thus, X_6 is a normalization of $X_2 \times_{\mathbb{C}^2} X_3$, and the diagram $(*_3)$ describes a normalization of the fibre product in a neighbourhood of a sp-point of type A_2 .

The diagram $(*_3)$ can be visually-schematic presented as follows

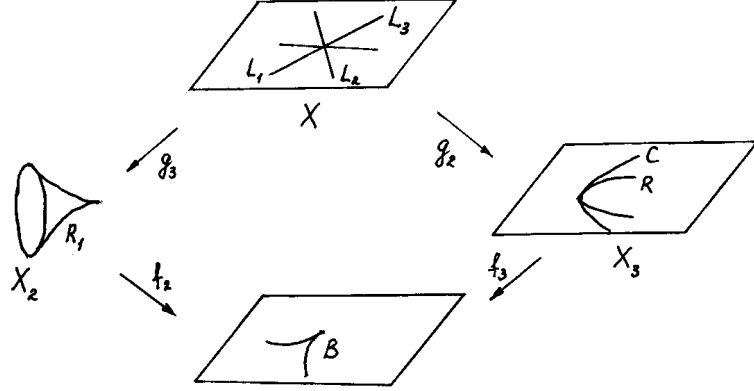


Fig. 7

Direct computations show that $x - z^2 = \frac{1}{3}(z_2 - z_1)(z_1 - z_3)$, and $x - \frac{1}{4}z^2 = \frac{1}{12}(z_3 - z_2)^2$, i.e.

$$g_2^*(R) = L_2 + L_3, \quad g_2^*(C) = 2L_1.$$

And, besides, $g_3^*(R_1) = L_1 + L_2 + L_3$.

2.4.4 Description of a normalization of the fibre product in a neighbourhood of a pp-point of type A_2 . Let $x_1 \in X_1$ and $x_2 \in X_2$ be p-points of type A_2 for f_1 and f_2 , $f_1^*(B) = 2R_1 + C_1$, $f_2^*(B) = 2R_2 + C_2$. In this case the 3-sheeted coverings f_1 and f_2 are the same (equivalent), and the monodromy homomorphisms $\varphi_1, \varphi_2 : \pi_1 = \pi_1(\mathbb{C}^2 \setminus B, y_0) \rightarrow S_3$ are epimorphic. The fibre $(f^\times)^{-1}(y_0)$ of the 9-sheeted covering $f^\times : X^\times = X_1 \times_{\mathbb{C}^2} X_2 \rightarrow \mathbb{C}^2$ consists of pairs $f_1^{-1}(y_0) \times f_2^{-1}(y_0) = \{(i, j), 1 \leq i, j \leq 3\}$, and the monodromy homomorphism is (equivalent to) a diagonal homomorphism $\varphi : \pi_1 \rightarrow S_3 \times S_3 \subset S_9$. Since φ_i are epimorphic, the fibre $(f^\times)^{-1}(y_0)$ consists of two orbits w.r.t. the action of π_1 — the orbit of the point $(1, 1)$, which consists of 3 elements, and the orbit of the point $(1, 2)$, which consists of 6 elements. From this and from Lemma 2.1 it follows that in a neighbourhood of the $\bar{x} = (x_1, x_2)$ a normalization X of the product X^\times consists of two disjoint components $X = X_3 \amalg X_6$, and on X_3 the morphism $f : X \rightarrow \mathbb{C}^2$ coincides with f_3 , the morphisms g_1 and g_2 are isomorphisms, and on X_6 the morphism $f = f_6$, the morphisms g_1 and g_2 are the same as g_2 in the diagram $(*_3)$

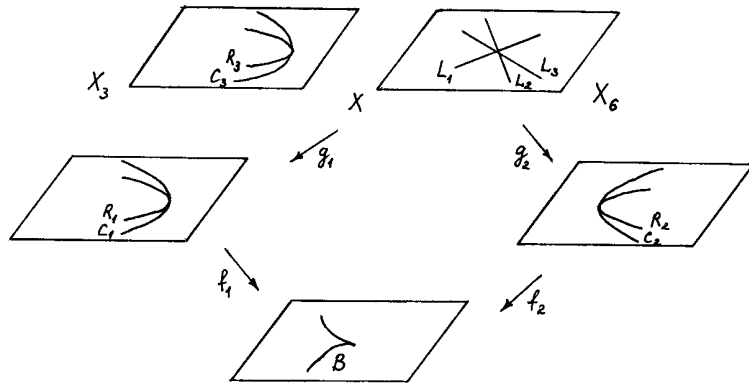


Fig. 8

There are 4 curves on X^\times : $C_1 \times_B C_2$, $R_1 \times_B C_2$, $C_1 \times_B R_2$, $R_1 \times_B R_2$, preimages of which on the normalization X are C_3 , L_2 , L_1 , and L_3 , R_3 . Under such a numeration of the lines L_i we have

$$g_1^*(R_1) = R_3 + L_2 + L_3, \quad g_1^*(C_1) = C_3 + 2L_1,$$

$$g_2^*(R_2) = R_3 + L_1 + L_3, \quad g_2^*(C_2) = C_3 + 2L_2.$$

2.4.5 *A lift of the diagram $(*_3)$.* Consider the diagram $(*_3)$. For computation of intersection numbers in §5 we need to resolve the singular point of type A_2 on the surface X_2 , and to ‘disjoint’ the curves L_2 and L_3 on X . A resolution of the singular point of type A_2 , as of any ‘double plane’, can be obtained, if we firstly take an imbedded resolution $\sigma : \bar{\mathbb{C}}^2 \rightarrow \mathbb{C}^2$ of the branch curve $B \subset \mathbb{C}^2$, and then take a normalization of $X_2 \times_{\mathbb{C}^2} \bar{\mathbb{C}}^2$.

Actually we’ll make more – we lift the whole of the diagram $(*_3)$ on $\bar{\mathbb{C}}^2$.

1) The singular point of B is resolved by one σ -process σ_1 . It is enough for the resolving of the singular point on X_2 , but to resolve the total transform of B up to a divisor with normal crossings, one need two more σ -processes. We picture the resolution process schematically by ‘drawing’ the total transform of the curve B . Denote by E_i the curve glued in under the i -th σ -process, and also its proper transform under subsequent σ -processes.

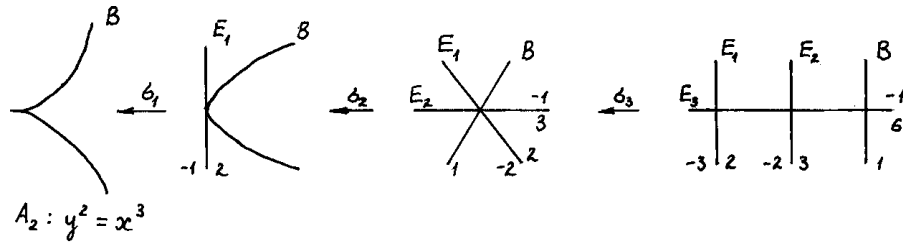
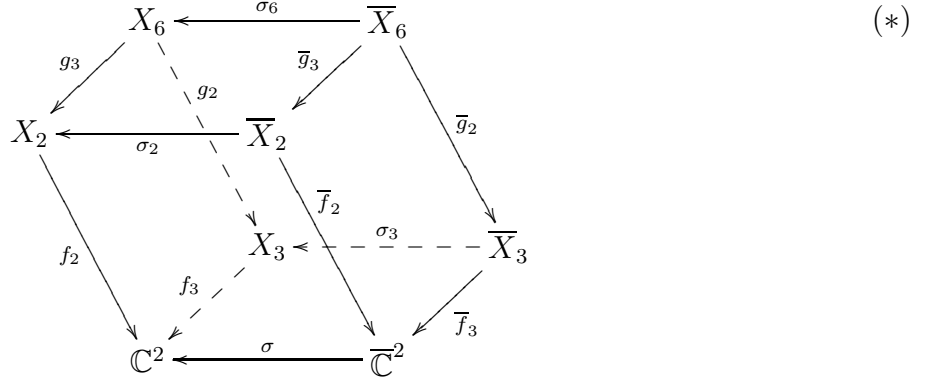


Fig. 9

Along each curve we indicate two numbers: the negative is the self-intersection number, the positive is its multiplicity in the total transform of the curve B .

2) Denote by $\sigma : \bar{\mathbb{C}}^2 \rightarrow \mathbb{C}^2$ the composition $\sigma_3 \circ \sigma_2 \circ \sigma_1$. We add on the diagram $(*_3)$ over $\bar{\mathbb{C}}^2$ and obtain a diagram as follows, on which all morphisms on the right face are finite coverings.



The right square of the diagram $(*)$ is obtained as a fibre product $(*_3) \times_{\mathbb{C}^2} \bar{\mathbb{C}}^2$, i.e. \bar{X}_i are normalizations of $X_i \times_{\mathbb{C}^2} \bar{\mathbb{C}}^2$, and morphisms are induced by morphisms of the diagram $(*_3)$ and projections. We describe how one can construct the diagram $(*)$ not uniformly as a normalization of the lift, but step-by-step. To facilitate the following of the description we begin with the final picture. We draw the right square of the diagram $(*)$ by replacing the varieties at its vertices by the total transforms of the curve B

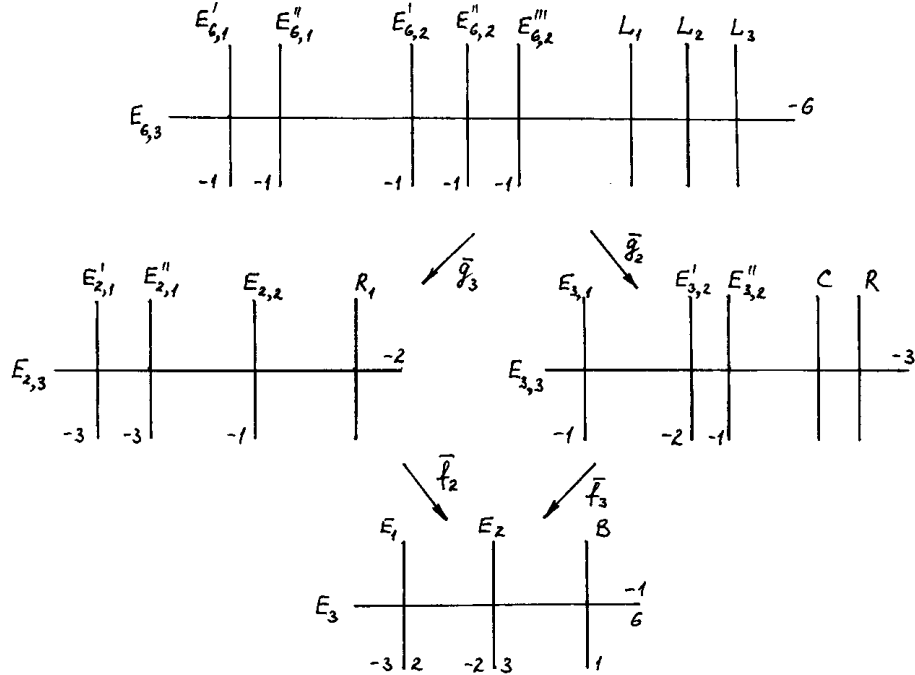


Fig. 10

The rule of notation is as follows. The exceptional curves E_1, E_2, E_3 on $\bar{\mathbb{C}}^2$ are already denoted. Under double indexing $E_{i,j}$ the first index indicates the variety X_i , where $E_{i,j}$ lies, and the second index indicates to what curve E_j the curve $E_{i,j}$ is mapped on $\bar{\mathbb{C}}^2$.

3) We begin a description of the diagram $(*)$ with \bar{X}_6 ('from the top'). To disjoint the lines L_i , we make σ -process with centre at the point $0 \in X_6 = \mathbb{C}^2 = \{z_1 + z_2 + z_3 = 0\} \subset \mathbb{C}^3$. By this the curve $E_{6,3} = \mathbb{P}^1 = \{t_1 + t_2 + t_3 = 0\} \subset \mathbb{P}^2 \ni (t_1 : t_2 : t_3)$ is glued, and we obtain a variety X'_6 . The action of S_3 on X_6 is extended to X'_6 and, in particular, to \mathbb{P}^1 . On \mathbb{P}^1 there are 8 exceptional points forming exceptional orbits:

$$p_1 = E_{6,3} \cap L_1 = (-2 : 1 : 1), p_2 = E_{6,3} \cap L_2 = (1 : -2 : 1), p_3 = E_{6,3} \cap L_3 = (1 : 1 : -2);$$

$$P_1 = (0 : 1 : -1), P_2 = (1 : 0 : -1), P_3 = (1 : -1 : 0);$$

$$Q_1 = (1 : \zeta : \zeta^2), Q_2 = (1 : \bar{\zeta} : \bar{\zeta}^2),$$

where $\zeta = \sqrt[3]{1}$ is a primitive root, and $\bar{\zeta} = \zeta^2$. Denote by $\xi = (123)$ a generator of the cyclic group of order three $\mathbb{Z}_3 = \mathcal{A}_3 = \{(1), (123), (132)\} \subset S_3$, and by $\varepsilon = (23)$ a generator of the cyclic group of order two $\mathbb{Z}_2 = S_2 = \{(1), (23)\} \subset S_3$. Then

$$\xi(p_1) = p_2, \xi(p_2) = p_3, \xi(p_3) = p_1; \quad \xi(P_1) = P_2, \xi(P_2) = P_3, \xi(P_3) = P_1;$$

$$\xi(Q_1) = (\zeta^2 : 1 : \zeta) = (1 : \zeta : \zeta^2) = Q_1, \xi(Q_2) = (\zeta : 1 : \zeta^2) = (1 : \zeta^2 : \zeta) = Q_2;$$

$$\varepsilon(p_1) = p_1, \varepsilon(p_2) = p_3, \varepsilon(p_3) = p_2; \quad \varepsilon(P_1) = P_1, \varepsilon(P_2) = P_3, \varepsilon(P_3) = P_2; \quad \varepsilon(Q_1) = Q_2.$$

If we take a quotient X'_6 under the action of $\mathbf{Z}_3 = \mathcal{A}_3$, then the stationary points Q_1 and Q_2 give two quotient singularities on $X'_2 = X'_6/\mathcal{A}_3$, resolving of which $X''_2 \rightarrow X'_2$ glues the curves $E'_{2,1}$ and $E''_{2,1}$ with $(E'_{2,1})^2 = -3$, $(E''_{2,1})^2 = -3$. To lift $X'_6 \rightarrow X'_2$ onto X''_2 , we have to blow up the points Q_1 and Q_2 , $X'_6 \rightarrow X'_6$, and by this we obtain $X''_6 \rightarrow X''_2$,

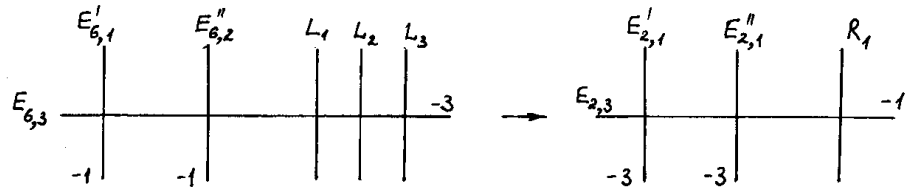


Fig. 11

4) The map f_2 is a factorization under the cyclic group $\mathbb{Z}_2 = S_2$. The action extends to X''_2 . The stationary point on $E_{2,3}$ – the image of the point P_1 on $E_{6,3}$ gives a singular point of type A_2 on X''_2/\mathbb{Z}_2 ($= \bar{\mathbb{C}}^{2'}$). A resolution of this point glues a (-2) -curve E_2 , and we obtain $\bar{\mathbb{C}}^2$. To lift $X''_2 \rightarrow X''_2/\mathbb{Z}_2$ onto the resolution $\bar{\mathbb{C}}^2$, we have to blow up a point on X''_2 . By this a (-1) -curve $E_{2,2}$ is glued, and we obtain \bar{X}_2 . To obtain $\bar{g}_3 : \bar{X}_6 \rightarrow \bar{X}_2$, we have to perform 3

σ -processes with centres at points P_1, P_2, P_3 on X''_6 , by which three lines $E'_{6,2}, E''_{6,2}$ and $E'''_{6,2}$ are glued. We obtain the left side \bar{g}_3 and \bar{f}_2 of the right square of diagram (*), pictured on Fig. 10. Note that the map \bar{g}_3 is ramified along the curves $E'_{6,1}$ and $E''_{6,1}$, and the map \bar{f}_2 is ramified along the curves $E_{2,2}$ and R_1 .

We can blow down the (-1) -curve $E_{2,2}$ on \bar{X}_2 , and then to blow down the (-1) -curve $E_{2,3}$. By this we obtain a minimal resolution of the singular point of type A_2 on X_2 ,

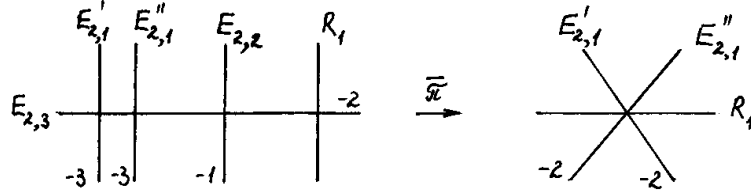


Fig. 12

5) The map \bar{g}_2 is a factorization under the group $\mathbf{Z}_2 = S_2 = \{(1), (23)\}$. We obtain the surface $\bar{X}_3 = \bar{X}_6/S_2$, $\bar{g}_2 : \bar{X}_6 \rightarrow \bar{X}_3$. The map \bar{g}_2 is ramified along the curves $E'_{6,2}$ and L_2 , which are mapped onto $E'_{3,2}$ and C correspondingly. The diagram is completed by the map $\bar{f}_3 : \bar{X}_3 \rightarrow \mathbb{C}^2$. The surface \bar{X}_3 is obtained from $X_3 = \mathbb{C}^2$, if we at first blow up the point of tangency of curves C and R gluing $E'_{3,2}$; then we blow up the point of intersection of C and R gluing $E_{3,3}$; finally, we blow up two more points on $E_{3,3}$:

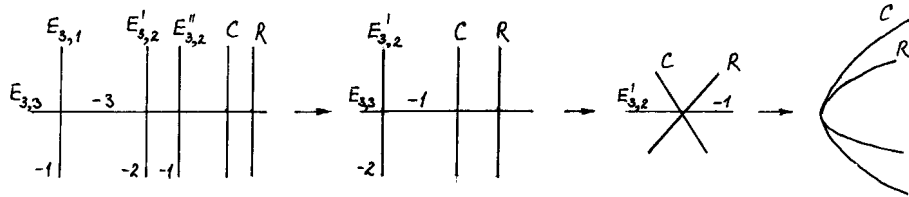


Fig. 13

3 The canonical cycle of a Du Val singularity

We intend to apply Hodge index theorem to obtain the basic inequality for generic coverings of \mathbb{P}^2 by surfaces with A-D-E-singularities. For this we need intersection theory and, therefore, a resolution of singularities of X . In this section we examine the local situation and find out how the resolution affects the canonical class and the ramification curve.

3.1. Definition of canonical cycle. Let (X, x) be a 2-dimensional A-D-E-singularity. Let $\pi : \bar{X} \rightarrow X$ be a minimal resolution, $L = \pi^{-1}(x)$ be the exceptional curve. As is known, the

canonical class $K_{\bar{X}}$ is trivial in a neighbourhood of L , that is we can choose a divisor in $K_{\bar{X}}$ with a support not intersecting L . In other words, there is a differential form ω on \bar{X} , which has neither poles nor zeroes in a neighbourhood of L . Such a form can be obtained, for example, as follows. As is known, (X, x) is a quotient singularity, $X = \mathbb{C}^2/G$, where $G \subset SL(2, \mathbb{C})$. The form $du \wedge dv$ on $\mathbb{C}^2 \ni (u, v)$ is invariant w.r.t. G and it defines a form on X ($\varphi^*(\omega) = du \wedge dv$, where $\varphi : \mathbb{C}^2 \rightarrow X$). Hence, the divisor $(\omega) = \sum k_i L_i$. Since L_i are (-2) -curves, $(L_i \cdot (\omega)) = 0$, and we obtain $(\omega) = 0$.

On the other hand, (X, x) can be considered as a double plane, that is as a 2-sheeted covering $X \xrightarrow{f} Y$ of the plane $Y = \mathbb{C}^2$ (locally). Let $z^2 = h(x, y)$ be an equation of (X, x) , $B : h(x, y) = 0$ be the discriminant curve, $f^{-1}(B) = R$, defined by the equation $z = 0$, be the ramification curve. We can consider the differential form $\omega = f^*(dx \wedge dy)$ lifted from Y . Then on \bar{X} the divisor $(\omega) = (z) = \bar{R} + Z$, where $\bar{R} \subset \bar{X}$ is the proper transform of R , $Z = \sum \gamma_i L_i$ is a cycle on $L = \pi^{-1}(x)$. We shall say that Z is the *canonical cycle* of a 2-dimensional A-D-E-singularity. Thus, $-Z$ is a cycle on the exceptional curve L , which is equivalent to the ramification curve \bar{R} in a neighbourhood of L . Let us calculate the canonical cycle for all A-D-E-singularities.

3.2. On resolution of double planes. As for any double plane, a resolution of an A-D-E-singularity can be obtained by means of a resolution of the discriminant curve $B \subset Y = \mathbb{C}^2$, $B : h(x, y) = 0$. Let $\sigma : \bar{Y} \rightarrow Y$ be a composition of σ -processes, such that the total transform of B is a divisor with normal crossings. Let $\sigma^*(B) = \bar{B} + \sum_{i=1}^r \alpha_i l_i$, where \bar{B} is the proper transform of B , $l_i \simeq \mathbb{P}^1$, $i = 1, \dots, r$, are the exceptional curves, as well as their proper transforms, glued by σ -processes. Let \bar{X} be the normalization of $\bar{Y} \times_Y X$, and \bar{f} and π be induced by projections,

$$\begin{array}{ccc} \pi^{-1}(x) = L = L_1 \cup \dots \cup L_r \subset \bar{X} & \xrightarrow{\pi} & X \supset R \ni x, R : z = 0 \\ \bar{f} \downarrow & & \downarrow f \\ \sigma^*(B) = \bar{B} + \sum_{i=1}^r \alpha_i l_i \subset \bar{Y} & \xrightarrow{\sigma} & Y \supset B. \end{array} \quad (\#)$$

Set $\bar{f}^{-1}(l_i) = L_i$. The curve L_i is either irreducible or consists of two components $L_i = L'_i + L''_i$, where $L'_i \simeq \mathbb{P}^1$, $L''_i \simeq \mathbb{P}^1$. The mapping \bar{f} is a 2-sheeted covering branched along the curve $\bar{B} + \sum_{\alpha_i \text{ odd}} l_i$. To be more graphic we denote the curves l_i , for which α_i are odd, also by \bar{l}_i , and L_i – respectively by \bar{L}_i . The surface \bar{X} has singularities of type A_1 over nodes of the branch curve $\bar{B} + \sum \bar{l}_i$. If this curve is non-singular, that is, a disconnected union of components (one can reach this by performing one additional σ -processes for each node), then \bar{X} is non-singular and is a resolution of the singularity (X, x) . Let \bar{R} be the proper transform of R w.r.t. π (= the proper transform of \bar{B} w.r.t. \bar{f}). We have $\bar{f}^*(\bar{l}_i) = 2\bar{L}_i$, if α_i is odd, and $\bar{f}^*(l_i) = L_i$, if α_i is even. We have

$$((\sigma \circ \bar{f})^* h(x, y)) = (z^2) = 2\bar{R} + \sum_{\alpha_i \text{ odd}} 2\alpha_i \bar{L}_i + \sum_{\alpha_i \text{ even}} \alpha_i L_i$$

and, consequently, $(z) = \bar{R} + Z$, where

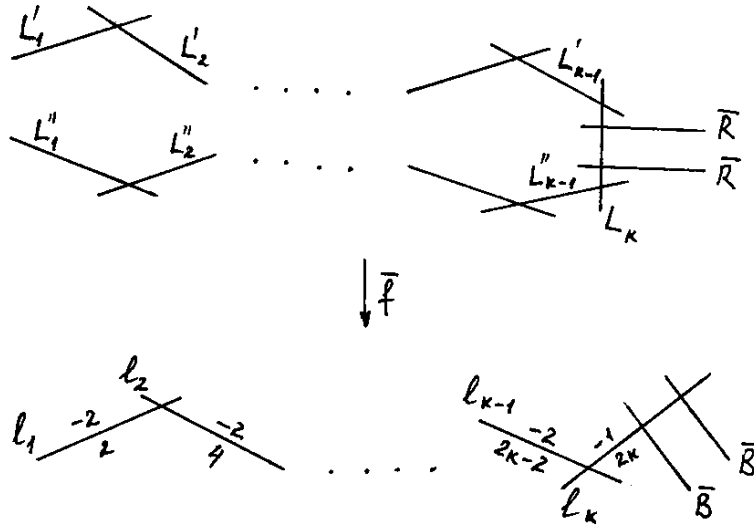
$$Z = \sum_{\alpha_i \text{--odd}} \alpha_i \bar{L}_i + \sum_{\alpha_i \text{--even}} \frac{1}{2} \alpha_i L_i .$$

Let us compute the cycle Z for each type of A-D-E-singularities (despite of abundance of papers concerning Du Val singularities, the authors do not know any of them, where the cycle Z is written out; so we have to perform these computations).

3.3. Computation of the canonical cycle. Consider the minimal resolution of each type of A-D-E-singularities described above. The following lemma contains the results of computations of $\sigma^*(B)$, of the exceptional curve $\pi^{-1}(x) = L$ and of the canonical cycle Z .

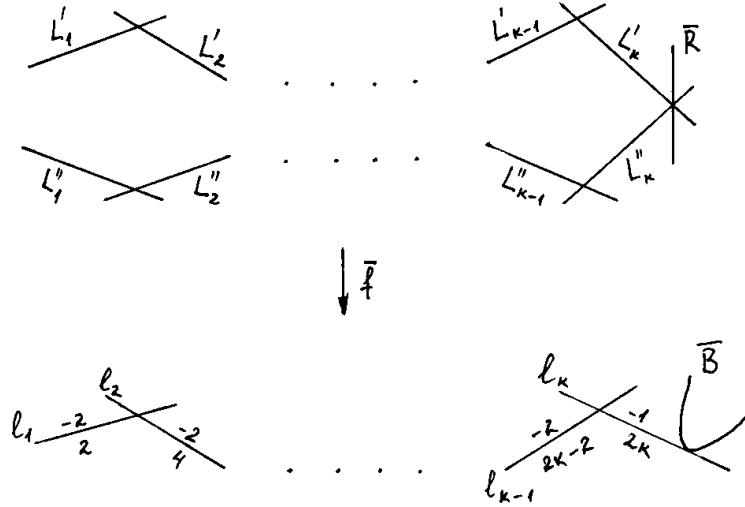
Lemma 3.1 *Below we picture schematically the total transform $\sigma^*(B) = \bar{B} + \sum_{i=1}^r \alpha_i l_i$ (near each curve l_i a positive number α_i and a negative number (l_i^2) are shown), and over it we picture the curve $\pi^{-1}(R)$, consisting of \bar{R} and (-2) -curves, and besides we write down the canonical cycle Z :*

1) The singularity $A_{2k-1} : y^2 = x^{2k}$, $k \geq 1$,



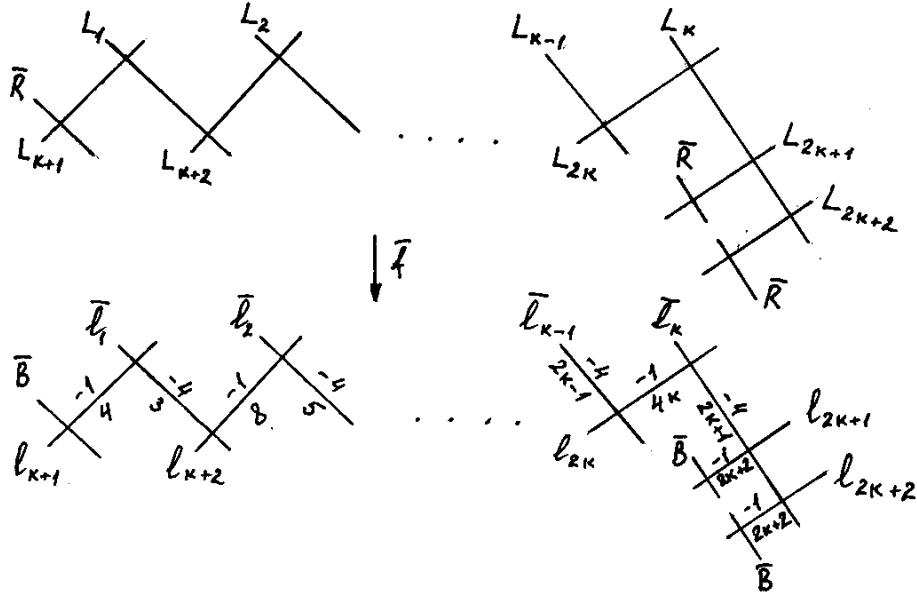
$$Z = L_1 + 2L_2 + \dots + kL_k ;$$

2) The singularity $A_{2k} : y^2 = x^{2k+1}$, $k \geq 1$,



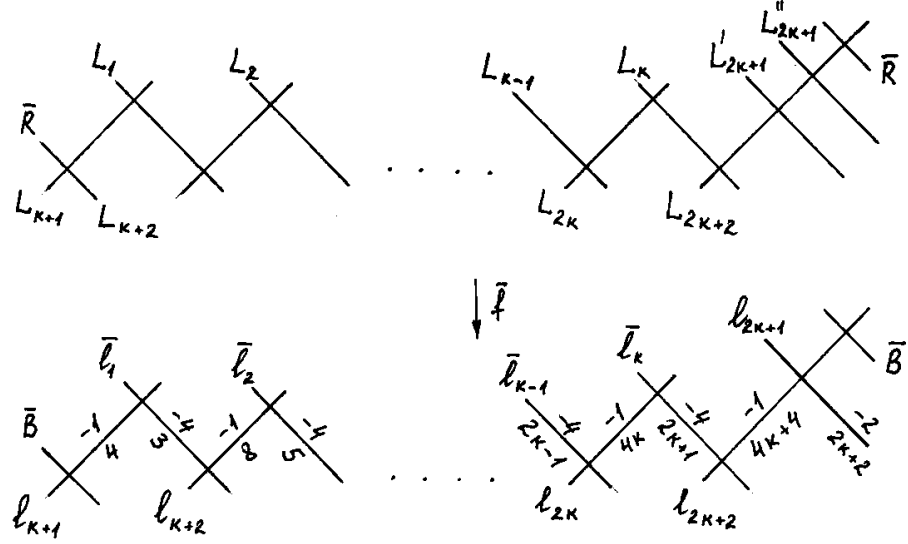
$$Z = L_1 + 2L_2 + \dots + kL_k;$$

3) The singularity $D_{2k+2} : x(y^2 + x^{2k})$, $k \geq 1$,



$$Z = 3L_1 + 5L_2 + \dots + (2k+1)L_k + 2L_{k+1} + 4L_{k+2} + \dots + 2kL_{2k} + (k+1)L_{2k+1} + (k+1)L_{2k+2};$$

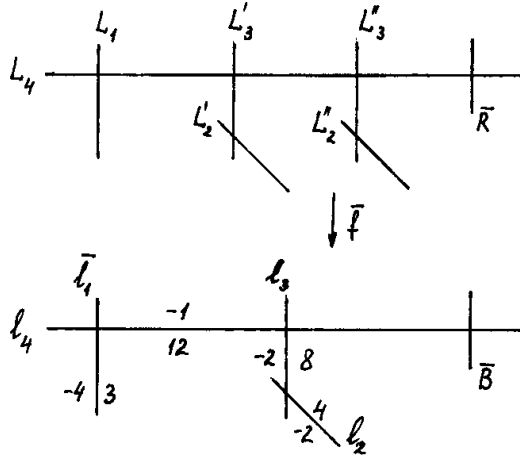
4) The singularity $D_{2k+3} : x(y^2 + x^{2k+1})$, $k \geq 1$,



$$Z = 3L_1 + 5L_2 + \dots + (2k+1)L_k + 2L_{k+1} + 4L_{k+2} + \dots + 2kL_{2k} + (2k+2)L_{2k+2} + (k+1)L_{2k+1};$$

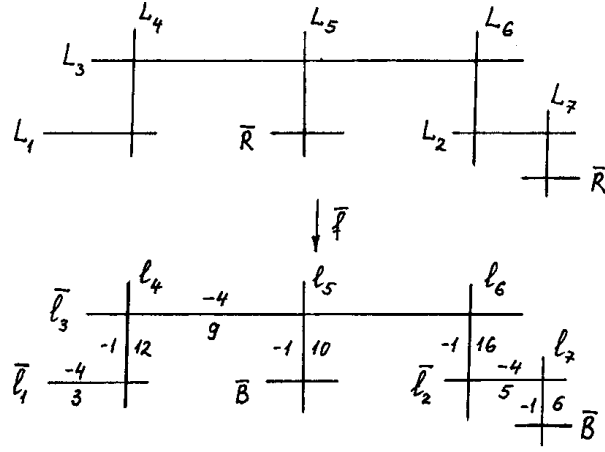
$$L_{2k+1} = L'_{2k+1} + L''_{2k+1};$$

5) The singularity $E_6 : x^3 + y^4$,



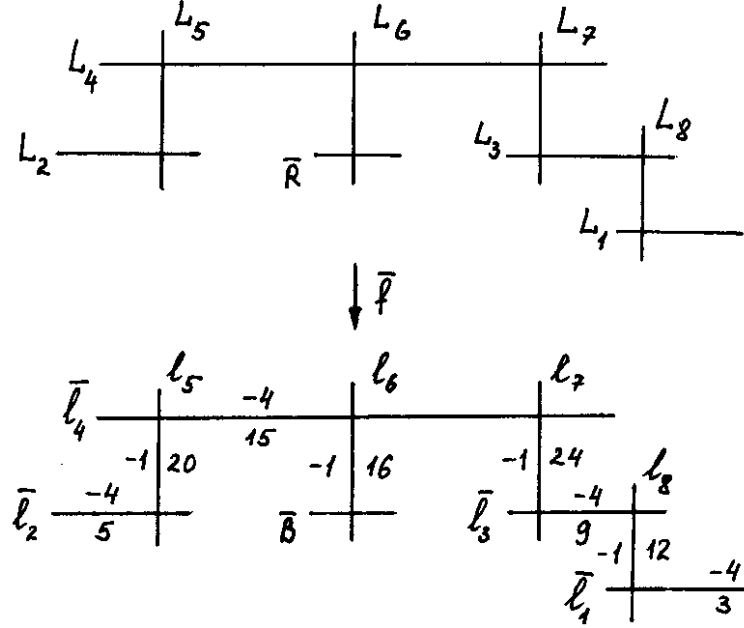
$$Z = 3L_1 + 2L_2 + 4L_3 + 6L_4;$$

6) The singularity $E_7 : x(x^2 + y^3)$,



$$Z = 3L_1 + 5L_2 + 9L_3 + 6L_4 + 5L_5 + 8L_6 + 3L_7;$$

7) The singularity $E_8 : x^3 + y^5$,



$$Z = 3L_1 + 5L_2 + 9L_3 + 15L_4 + 10L_5 + 8L_6 + 12L_7 + 6L_8. \quad \blacksquare$$

3.4. Defect of a singularity. Define a defect δ of a A-D-E-singularity by the formula

$$\delta = \frac{1}{2}(\bar{R} \cdot Z).$$

Corollary 3.1 *For different types of A-D-E-singularities the defect equals*

$$\delta = \begin{cases} \left\lfloor \frac{n+1}{2} \right\rfloor & \text{for type } A_n; \\ \left\lfloor \frac{n}{2} \right\rfloor + 1 & \text{for type } D_n; \\ \left\lfloor \frac{n+1}{2} \right\rfloor & \text{for types } E_n, n = 6, 7, 8. \end{cases}$$

In particular, for the type A_1 (nodes) and A_2 (cusps) the defect $\delta = 1$.

Actually one can show that defect δ is the δ -invariant (genus) of the one-dimensional A-D-E-singularity.

4 Numerical invariants of a generic covering

Now we consider a global situation. Let X be a surface with A-D-E-singularities,

$$\text{Sing } X = \sum_{k \geq 1} a_k A_k + \sum_{k \geq 4} d_k D_k + \sum_{k=6,7,8} e_k E_k,$$

that means that X has a_k singularities of type A_k , d_k – of type D_k and e_k – of type E_k . Let $f : X \rightarrow \mathbf{P}^2$ be a generic covering of degree N , and $B \subset \mathbf{P}^2$ be the discriminant curve. Let $\deg B = d$ and let B has n nodes and c cusps, $n_s = a_1$ and $c_s = a_2$ of which originates from $\text{Sing } X$, and n_p and c_p are p-nodes and p-cusps. Let $R \subset X$ be the ramification curve, $f^*(B) = 2R + C$, and $L \subset X$ be the preimage of a generic line $l \subset \mathbf{P}^2$. Let $\pi : S \rightarrow X$ be the minimal resolution of X , and $\bar{f} = f \circ \pi : S \rightarrow \mathbf{P}^2$. Denote by \bar{R} and \bar{L} the proper transforms of R and L on S . Then \bar{R} is a normalization of the curve $R \simeq B$, and $\bar{L} \simeq L$.

4.1. *The canonical class K_S and the canonical cycle Z . Let*

$$Z = \sum_{x \in \text{Sing } X} Z_x$$

be the canonical cycle of S , where Z_x are the canonical cycles of singularities $x \in \text{Sing } X$. It follows from 3.2 that

$$K_S = (f \circ \pi)^* K_{\mathbf{P}^2} + \bar{R} + Z = -3\bar{L} + \bar{R} + Z. \quad (4.1)$$

Besides, the singularities of X being Gorenstein, the divisor R is locally principal, and

$$\pi^*(R) = \bar{R} + Z. \quad (4.2)$$

4.2. *The intersection numbers.*

Lemma 4.1 *The intersection numbers of \bar{L} , \bar{R} and Z on S are equal*

$$(\bar{L}^2) = N, \quad \bar{L} \cdot \bar{R} = d, \quad \bar{L} \cdot Z = 0, \quad \bar{R} \cdot Z = 2\delta_X, \quad (Z^2) = -2\delta_X, \quad (4.3)$$

where

$$\delta_X = \sum_{x \in \text{Sing } X} \delta_x = \sum a_k \left\lfloor \frac{k+1}{2} \right\rfloor + \sum d_k \left(\left\lfloor \frac{k}{2} \right\rfloor + 1 \right) + \sum e_k \left\lfloor \frac{k+1}{2} \right\rfloor \quad (4.4)$$

is the defect of the surface X .

Proof. Obviously, we have $(\bar{L}^2) = \deg f = N$, and $\bar{L} \cdot \bar{R} = \deg B = d$. By 3.4 we have $\bar{R} \cdot Z = 2\delta_X$. The divisor Z being exceptional, we have $\pi(Z) = \text{Sing } X$, $\dim \pi(Z) = 0$, and $\bar{L} = \pi^*(L)$, $\bar{R} + Z = \pi^*(R)$, and therefore, $\bar{L} \cdot Z = 0$, and $(\bar{R} + Z) \cdot Z = 0$, and, consequently, $(Z^2) = -(\bar{R} \cdot Z)$. ■

It remains to compute (\bar{R}^2) .

4.3. *The evenness of degree* $\deg B = d = 2\bar{d}$. The restriction of \bar{f} to \bar{L} , $\bar{L} \rightarrow l \simeq \mathbb{P}^1$, is a covering of degree N , with ramification indices 2 at the points of intersection of \bar{L} and \bar{R} . We have $\bar{L} \cdot \bar{R} = d$, and from Hurwitz formula we obtain $2g(\bar{L}) - 2 = -2N + d$. It follows that $\deg B = d$ is even. Let $d = 2\bar{d}$. Besides, since

$$g(\bar{L}) = \frac{1}{2}d + 1 - N \geq 0,$$

we obtain a bound for the degree of covering,

$$N \leq \bar{d} + 1.$$

4.4. *The self-intersection number (\bar{R}^2) and the arithmetical genus of the curve R .* Denote by δ the defect of the curve B ,

$$\delta = \delta_B = \sum_{s \in \text{Sing } B} \delta_s = n + c + \delta_0, \quad (4.5)$$

where

$$\delta_0 = \sum_{x \in \text{Sing } B, x \text{ not } A_1 \text{ and } A_2} \delta_x. \quad (4.6)$$

The numbers δ and δ_0 are the extremal values of defects δ_X of surfaces X with given discriminant curve B : δ_0 corresponds to a surface X , all nodes and cusps of which are p-nodes and p-cusps, $n = n_p$, $c = c_p$, and δ corresponds to a surface X (for example, a 2-sheeted covering of \mathbb{P}^2), all nodes and cusps of which are s-nodes and s-cusps, $n = n_s$, $c = c_s$.

At first we express the geometric genus of B , $g = g(B) = g(\bar{R})$, in terms of the defect δ . For this we consider a surface X , which is a 2-sheeted covering of \mathbb{P}^2 with the discriminant curve B . In this case $(Z^2) = -(\bar{R} \cdot Z) = -2\delta$, and $f^*(B) = 2R$ and, consequently, $d \cdot \bar{L} \sim 2\bar{R} + 2Z$, because $B \sim d \cdot l$. From (4.1) and the adjunction formula $g(\bar{R}) = \frac{(\bar{R}, \bar{R} + K_{\bar{X}})}{2} + 1$ we obtain

$$g = \frac{(d-1)(d-2)}{2} - \delta. \quad (4.7)$$

If it is known that the defect δ coincides with the δ -invariant of a one-dimensional singularity, then this formula coincides with the well known formula for the geometric genus $g(R) \stackrel{df}{=} g(\bar{R})$ of a singular curve R , $g(R) = p_a(R) - \sum_{x \in \text{Sing } R} \delta_x$.

We return to a generic covering X of degree N , $n = n_s + n_p$, $c = c_s + c_p$. Then

$$\delta_X = n_s + c_s + \delta_0 = \delta - n_p - c_p. \quad (4.8)$$

Lemma 4.2 *The self-intersection number of the proper transform of the ramification curve $\bar{R} \subset S$ is equal*

$$(\bar{R}^2) = 3\bar{d} + g - 1 - \delta_X, \quad (4.9)$$

and

$$(\bar{R} + Z)^2 = 3\bar{d} + g - 1 + \delta_X = 3\bar{d} + p_a(R) - 1, \quad (4.10)$$

where

$$p_a(R) = g + \delta_X = \frac{(d-1)(d-2)}{2} - n_p - c_p \quad (4.11)$$

is the arithmetical genus of R .

Proof. From (4.1) and the adjunction formula $2g(\bar{R}) - 2 = (\bar{R}, \bar{R} + K_S) = (\bar{R}, -3\bar{L} + 2\bar{R} + Z)$ we obtain $(\bar{R}^2) = \frac{3}{2}(\bar{R} \cdot \bar{L}) + g - 1 - \frac{1}{2}(\bar{R} \cdot Z)$. Applying formulae (4.3), we obtain the proof. ■

From formulae (4.1), (4.3) and (4.9) we obtain a corollary.

Corollary 4.1

$$(K_S^2) = 9N - 9\bar{d} + p_a(R) - 1, \quad (4.12)$$

or, substituting $p_a(R)$ from (4.11),

$$(K_S^2) = 9N + \frac{1}{2}d(d-12) - n_p - c_p. \quad (4.12')$$

4.5. A bound for the covering degree.

Lemma 4.3 *For a generic covering of degree N with discriminant curve of degree $d = 2\bar{d}$ and genus g , we have*

$$N \leq \frac{4\bar{d}^2}{3\bar{d} + g - 1 + \delta_X}, \quad (4.13)$$

where δ_X is the defect of singularities of X , and moreover, the equality holds if and only if $\bar{L} \equiv mK_S$ for some $m \in \mathbb{Q}^*$, or $mK_S \equiv 0$.

Proof. Applying Hodge index theorem to divisors \bar{L} and $\pi^*(R) = \bar{R} + Z$ on S , we obtain

$$\begin{vmatrix} \bar{L}^2 & (\bar{L}, \bar{R} + Z) \\ (\bar{L}, \bar{R} + Z) & (\bar{R} + Z)^2 \end{vmatrix} = \begin{vmatrix} N & d \\ d & 3\bar{d} + g - 1 + \delta_X \end{vmatrix} \leq 0,$$

and it is the desired inequality. The equality holds only if \bar{L} and $\bar{R} + Z$ are linear dependent in the Néron-Severi group $NS(\bar{X}) \otimes \mathbb{Q}$. Since $K_S = -3\bar{L} + \bar{R} + Z$, we obtain the assertion about possible equality. ■

4.6. The topological Euler characteristic $e(S)$.

Lemma 4.4 *The topological Euler characteristic of a surface S , obtained by the minimal resolution of singularities of X , is connected with the defect δ_X and invariants of a generic covering f by a formula*

$$e(S) = 3N + 2g - 2 + 2\delta_X - c_p, \quad (4.14)$$

where $N = \deg f$, and c_p is the number of p -cusps on B (or the number of pleats of f).

Proof is obtained in the same way as in the case of a non-singular surface X ([K], §1 Lemma 7), considering a generic pencil of lines on \mathbb{P}^2 and the corresponding hyperplane sections on S , and lifting the morphism $\bar{f} : S \rightarrow \mathbb{P}^2$ to a morphism of fiberings of curves over \mathbb{P}^1 . One can obtain a proof by direct computations. At first we find $e(X) = 3N - e(B) - n_p - c_p$ by considering the finite covering $f : X \rightarrow \mathbb{P}^2$, the stratification $\mathbb{P}^2 = (\mathbb{P}^2 \setminus B) \cup (B \setminus \text{Sing } B) \cup \text{Sing } B$, and applying the additivity property of Euler characteristic, and then we find $e(S)$. ■

From Noether's formula $(K_S^2) + e(S) = 12p_a$ and formulae (4.12) and (4.14) we have $12p_a = 12N - 9\bar{d} + 3p_a(R) - 3 - c_p$. Substituting $p_a(R)$ from (4.11), we obtain a corollary.

Corollary 4.2 *The Euler characteristic of the structure sheaf \mathcal{O}_S equals*

$$p_a = 1 - q - p_g = N + \frac{\bar{d}(\bar{d} - 3)}{2} - \frac{n_p}{4} - \frac{c_p}{3}. \quad (4.15)$$

Thus, as in the case of a non-singular surface X , we obtain

Corollary 4.3

$$n_p \equiv 0 \pmod{4}, \quad c_p \equiv 0 \pmod{3}.$$

5 Proof of the main inequality.

5.1. A fiber product of two generic coverings. Let a curve B be a common discriminant curve for two generic coverings $f_1 : X_1 \rightarrow \mathbb{P}^2$ and $f_2 : X_2 \rightarrow \mathbb{P}^2$ of degrees $\deg f_1 = N_1$ and $\deg f_2 = N_2$. Let

$$\text{Sing } B = nA_1 + cA_2 + \sum_{k>2} a_k A_k + \sum_{k\geq 4} d_k D_k + \sum_{k=6,7,8} e_k E_k.$$

With respect to a pair of coverings f_1 and f_2 nodes and cusps of B are subdivided into four types,

$$n = n_{ss} + n_{sp} + n_{ps} + n_{pp}, \quad c = c_{ss} + c_{sp} + c_{ps} + c_{pp}, \quad (5.1)$$

where $n_{b\sharp}$ and $c_{b\sharp}$ are numbers of $b\sharp$ -nodes and $b\sharp$ -cusps of B . In particular, $n_{ss} + n_{sp} = a_1$ is the number of singularities of type A_1 , and $n_{ss} + n_{sp} = a_2$ is the number of singularities of type A_2 on the surface X_1 .

Consider a normalization X of the fiber product $X^\times = X_1 \times_{\mathbb{P}^2} X_2$ and the corresponding commutative diagram

$$\begin{array}{ccccc} & & X = \mathbb{C}^2 \ni (z_1, z_2) & & \\ & \swarrow g_1 & \downarrow g_0 & \searrow g_2 & \\ (z_1, y) \in \mathbb{C}^2 = X_1 & & X_0 & & X_2 = \mathbb{C}^2 \ni (x, z_2) \\ & \searrow f_1 & \downarrow f_0 & \swarrow f_2 & \\ & & (x, y) \in \mathbb{C}^2 & & \end{array} \quad (*_2)$$

The surface X is a $N_1 N_2$ -sheeted covering of \mathbb{P}^2 and it has at most A-D-E-singularities, which lie over $Sing B$.

Lemma. If coverings f_1 and f_2 are non equivalent, then the surface X is irreducible.

Proof is word for word the same as in the case of generic coverings of non-singular surfaces ([K] Proposition 2). ■

We set

$$g_1^{-1}(R_1) = R + C,$$

where R is a part, which is mapped by g_2 onto R_2 , and C is a part, which is mapped g_2 onto C_2 . We are interested in the intersection number of R and C after a resolution of singularities of X in a neighbourhood of the curve $R + C$.

Consider a restriction $R + C \rightarrow R_1$ of the covering g_1 over the curve R_1 . As follows from 2.2.1 and 2.2.2, it is an étale covering of degree N_2 over a generic point $x_1 \in R_1$, where $R \rightarrow R_1$ is a 2-sheeted, and $C \rightarrow R_1$ is a $(N_2 - 2)$ -sheeted covering. The same picture is over a point $x_1 \in R_1$, which is a s-point of X_1 , lying over a ss-point of B .

Denote by $\tilde{\pi} : S \rightarrow X$ a minimal resolution of singularities of X , and denote by \tilde{R} and \tilde{C} the proper transforms of R and C on S . Our goal is to calculate the intersection numbers (\tilde{R}^2) , $(\tilde{R} \cdot \tilde{C})$ and (\tilde{C}^2) , and also the analogous intersection numbers for divisors $\tilde{\pi}^{-1}(R) = \tilde{R} + Z_R$ and $\tilde{\pi}^{-1}(C) = \tilde{C} + Z_C$, where Z_R and Z_C are the sums of canonical cycles corresponding to singular points $x \in Sing X$ and lying on R and C respectively.

5.2. The structure of a fibre product over a neighbourhood of a singular point of the discriminant curve. Let $U \subset \mathbb{P}^2$ be a sufficiently small neighbourhood (in complex topology) of a point $b \in Sing B$. The preimage $f_1^{-1}(U)$ is a disjoint union of two parts, $f_1^{-1}(U) = V_1 \sqcup V'_1$, where V_1 is a part containing the ramification curve R_1 , and V'_1 is a part not containing R_1 and étale mapped to U . Analogously $f_2^{-1}(U) = V_2 \sqcup V'_2$. Then $f^{-1}(U)$ is a disjoint union of four open sets – of normalizations of fibre products $W = \overline{V_1 \times_U V_2}$, $W' = \overline{V_1 \times_U V'_2}$, $\overline{V'_1 \times_U V_2}$ and $\overline{V'_1 \times_U V'_2}$. And only W and W' meet the curve $g_1^{-1}(R_1)$. The open sets $W \subset X$ were studied in detail in §2. The surface X in the neighbourhood W is non-singular except the case of ss-points b . The open set W' consists of $N_2 - k$ components ($k = 2, 3$ or 4 depending on the type of the singular point b), which are mapped isomorphically onto V_1 . And W' does not meet R , and $W' \cap C$ consists of $N_2 - k$ components isomorphic to $V_1 \cap R_1$.

It follows from the investigation of the local structure of X in §2 that X and the curves R and C are of the following form over neighbourhoods of singular points $b \in Sing B$ of different types.

1) Over a ss-point b the neighbourhood W has 2, and W' has $N_2 - 2$ components, which are mapped isomorphically onto V_1 by the map g_1 . Correspondingly $R \cap W$ consists of two, and $C \cap W'$ consists of $(N_2 - 2)$ components isomorphic to $R_1 \cap V_1$.

2) Over a sp-point $b \in B$ of type A_1 the neighbourhood W' consists of $(N_2 - 4)$ components isomorphic to V_1 and having a singular point of type A_1 . Correspondingly C consists of $N_2 - 4$ nodal curves. The neighbourhood W consists of two components: see Fig. 4, where

$$R = R' + R'' , \text{ and } C = R' + R''$$

(it ought to change places of the left and right parts of Fig. 4, g_1 stands for g_2 , and g_2 – for

g_1). We see that in the neighbourhood W the curves R and C are non-singular and meet transversally in two points.

3) Over a ps-point $b \in B$ of type A_1 the neighbourhood $V_1 \subset X_1$ consists of two components and on each of them the map f_1 has a fold. The neighbourhood W' consists of disjoint union of $(N_2 - 2)$ pieces isomorphic to V_1 . The neighbourhood W consists of two components: see Fig. 4, on which

$$R = R' + R'' , \text{ and } C = \emptyset.$$

We see that on W the curve R is non-singular and does not meet C .

4) Over a pp-point $b \in B$ of type A_1 the neighbourhood $V_1 \subset X_1$ consists of two irreducible components and on each of them f_1 has a fold. The neighbourhood W' is non-singular and consists of $N_2 - 4$ components isomorphic to V_1 . The neighbourhood W is represented on Fig. 5, on which

$$R = R' + R'' + R''' + R'''', C = C' + C''.$$

We see that the curves R and C are non-singular and do not meet.

5) Over a sp-point $b \in B$ of type A_2 the neighbourhood V_1 has a singular point of type A_2 , and W' consists of $(N_2 - 3)$ components isomorphic to V_1 . The neighbourhood W is pictured on Fig. 7, on which

$$R = L_2 + L_3 , C = L_1.$$

We see that R has a double point, C is non-singular and intersect transversally each of the branches of R at the intersection point, and, consequently, $(R \cdot C) = 2$.

6) Over a ps-point $b \in B$ of type A_2 the neighbourhood V_1 is non-singular, and W' consists of $(N_2 - 2)$ components isomorphic to V_1 . The neighbourhood W is pictured on Fig. 7 (on which it ought to change places of the left and right parts, g_1 stands for g_2 , and g_2 - for g_3), where

$$R = L_2 + L_3 , C = \emptyset .$$

We see that R has a double point and does not meet C .

7) Over a pp-point $b \in B$ of type A_2 the neighbourhood W' consists of $N_2 - 3$ components isomorphic to V_1 . The neighbourhood W is pictured on Fig. 8, on which

$$R = R_3 + L_3 , C = L_2 .$$

We see that R is non-singular and meets with C transversally at one point.

From the obtained local description it follows that the surface X is non-singular at the points of intersection of R and C , and the intersection is not void only over the points $b \in B$ of types: over sp-points of type A_1 , where $(R \cdot C) = 2$, over sp-points of type A_2 , where $(R \cdot C) = 2$, and over pp-pointe of type A_2 , where $(R \cdot C) = 1$. Therefore,

$$(\tilde{R} \cdot \tilde{C}) = 2n_{sp} + 2c_{sp} + c_{pp}. \quad (5.3)$$

5.3. *A lift of the fibre product to a resolution of the discriminant curve.* To compute intersection numbers on S we consider firstly an auxiliary surface \tilde{X} , which is not a minimal

resolution of X , and then we ‘descend’ to S . Let $\sigma : \bar{\mathbb{P}}^2 \rightarrow \mathbb{P}^2$ be a composition of σ -processes resolving the curve B and needed to obtain a minimal resolution of a double plane singularities, lying over B (see §3), and, besides, let σ includes two additional σ -processes as in 2.4.5 for each cusp, which is not a ss-cusp. Consider a lift of the diagram $(*)_1$ to $\bar{\mathbb{P}}^2$, namely consider the diagram

$$\begin{array}{ccccc}
 & & S & & \\
 & \swarrow \tilde{\pi} & & \searrow \bar{\pi} & \\
 X & \xleftarrow{\pi} & \bar{X} & & \\
 \swarrow g_1 & & \searrow \bar{g}_1 & & \\
 X_1 & \xleftarrow{\pi_1} & \bar{X}_1 & & \\
 \swarrow f_1 & & \searrow \bar{f}_1 & & \\
 \mathbb{P}^2 & \xleftarrow{\sigma} & \bar{\mathbb{P}}^2 & & \\
 & \nwarrow f_2 & \nwarrow \bar{f}_2 & & \\
 & X_2 & \xleftarrow{\pi_2} & \bar{X}_2 &
 \end{array}
 \tag{5.4}$$

in which \bar{X}_i and \bar{X} are normalizations of $X_i \times_{\mathbb{P}^2} \bar{\mathbb{P}}^2$ and $X \times_{\mathbb{P}^2} \bar{\mathbb{P}}^2$. Then morphisms ‘on the right wall’ of diagram (5.4) are finite coverings. The surface \bar{X} is non-singular, and $\bar{\pi} : \bar{X} \rightarrow S$ blows down the ‘superfluous’ exceptional curves of the first kind. Let \bar{R}_1 be the proper transform of R_1 on \bar{X}_1 , and \bar{R} and \bar{C} (respectively \tilde{R} and \tilde{C}) be the proper transforms of R and C on \bar{X} (respectively on S). Then $\bar{g}_1^*(\bar{R}_1) = \bar{R} + \bar{C}$, and $\bar{R} \rightarrow \bar{R}_1$ and $\bar{C} \rightarrow \bar{R}_1$ are finite coverings of degree 2 and $N_2 - 2$ respectively, and \bar{R} and \bar{C} are disjoint. Therefore,

$$(\bar{R}^2) = 2 (\bar{R}_1^2), \quad (\bar{C}^2) = (N_2 - 2) (\bar{R}_1^2), \quad \bar{R} \cdot \bar{C} = 0.
 \tag{5.5}$$

Actually from 3) and 4) one can see that over ps- and pp-nodes b in a neighbourhood of $R+C$ the surface X is non-singular, the curves R and C are non-singular and disjoint. Therefore, one can suppose that ps- and pp-nodes on B are not blown up (and on the surface S there remain singular points, which lie over these nodes).

5.4. Computation of intersection numbers. First we find (\bar{R}_1^2) . Recall that by (4.9) we have on the minimal resolution \tilde{X}_1 of the surface X_1

$$(\tilde{R}_1^2) = 3\bar{d} + g - 1 - \delta_1,
 \tag{5.6}$$

where $\delta_1 = \delta_{X_1} = n_s + c_s + \delta_0$, and $n_s = n_{ss} + n_{sp}$ and $s = s_s + s_p$ are the numbers of singular points of type A_1 and A_2 on the surface X_1 .

Let $\pi_1 = \tilde{\pi}_1 \circ \bar{\pi}_1$, where $\tilde{\pi}_1 : \tilde{X}_1 \rightarrow X_1$ is a minimal resolution, and $\bar{\pi}_1 : \bar{X}_1 \rightarrow \tilde{X}_1$ is the blowing down of the “superfluous” exceptional curves. The surface \bar{X}_1 differs from the surface \tilde{X}_1 only over the cusps of B , which are not ss-cusps. Let $\bar{U} = \sigma^{-1}(U)$, and $\bar{V}_1 = \pi_1^{-1}(V_1)$, $\tilde{V}_1 = \tilde{\pi}_1^{-1}(V_1)$ be neighbourhoods of \bar{X}_1 and \tilde{X}_1 lying over \bar{U} and containing the proper transform of R_1 . Analogously \bar{V}_1' and \tilde{V}_1' .

For a sp-cusp $b \in B$ the blowing down $\bar{V}_1 \xrightarrow{\bar{\pi}_1} \tilde{V}_1$ is represented on Fig. 12. We see that $\bar{\pi}$ includes one σ -process with a centre R_1 . For ps- and pp-cusps $b \in B$ the blowing down $\bar{V}_1 \xrightarrow{\bar{\pi}_1} \tilde{V}_1$ is represented on Fig. 13 (where R stands for R_1 , and C stands for C_1). We see that it needs two σ -process with centres on R_1 to disjoint C_1 and R_1 . Therefore,

$$(\bar{R}_1^2) = (\tilde{R}_1^2) - c_{sp} - 2c_{ps} - 2c_{pp}. \quad (5.7)$$

Now we examine how the intersection numbers (\bar{R}^2) and (\bar{C}^2) change under the blowing down $\bar{\pi}$. For a neighbourhood $U \subset \mathbb{P}^2$ of a point $b \in \text{Sing } B$ set $\bar{W} = \pi^{-1}(W)$, $\bar{W}' = \pi^{-1}(W')$, $\tilde{W} = \tilde{\pi}^{-1}(W)$, $\tilde{W}' = \tilde{\pi}^{-1}(W')$. Then $\bar{g}_1^{-1}(\bar{V}_1) = \bar{W} \sqcup \bar{W}'$. We examine one after another the blowing down $\bar{\pi} : \bar{X} \rightarrow S$ in neighbourhoods $\bar{W} \sqcup \bar{W}' \subset \bar{X}$ separately for different types of singular points $b \in \text{Sing } B$ (the numbering of cases corresponds to the numbering of cases in 5.2).

2) For a sp-point b of type A_1 the neighbourhood \bar{W}' is a disjoint union of $(N_2 - 4)$ open sets isomorphic to \bar{V}_1 – to the minimal resolution of singular points of type A_1 . The neighbourhood W is represented on Fig. 4, and $\bar{\pi} : \bar{W} \rightarrow W$ is a blowing up of two points $R'^{\prime\prime} \cap R''^{\prime\prime}$ and $R''^{\prime\prime} \cap R''^{\prime\prime}$. Therefore, the blowing down $\bar{\pi} : \bar{W} \rightarrow \tilde{W} \simeq W$ increases (\bar{R}^2) and (\bar{C}^2) on 2 for one point b and, consequently, on $2n_{sp}$ for all points of this type.

5) For a sp-point b of type A_2 the neighbourhood \bar{W} is represented on the upper part of Fig. 10. It is obtained from the neighbourhood W , pictured on Fig. 7, by blowing up the point of intersection of lines L_1, L_2 and L_3 , and then by blowing up 5 points on the glued line $E_{6,3}$ and not lying on the proper transform of these lines. The blowing down $\bar{\pi} : \bar{W} \rightarrow \tilde{W} \simeq W$ is the converse procedure, i.e. the blowing down of five exceptional curves of the first kind, and then blowing down the curve $E_{6,3}$. In this case $R = L_2 + L_3$, and $C = L_1$. Since $(R^2) = (L_2^2) + 2(L_2 \cdot L_3) + (L_3^2)$ and $(L_2^2), (L_3^2)$ are diminished on 1, and L_2 and L_3 are no longer intersected after the σ -process with the centre at the point $L_2 \cap L_3$, the blowing down $\bar{\pi}$ increases (\bar{R}^2) on 4 for one point b and on $4c_{sp}$ for all points of this type.

The neighbourhood \bar{W}' consists of $(N_2 - 3)$ components isomorphic to \bar{V}_1 , for each of which $\bar{\pi}$ is represented on Fig. 12. As above for (\bar{R}_1^2) , we see that the blowing down $\bar{\pi}$ increases (\bar{C}^2) on $(N_2 - 3) + 1$ (taking account of the neighbourhood \bar{W}) for one point b and on $(N_2 - 2)c_{sp}$ for all points of this type.

6) For a ps-point b of type A_2 the neighbourhood \bar{W} and the blowing down $\bar{\pi} : \bar{W} \rightarrow \tilde{W} \simeq W$ are the same as in 5), but in this case $R = L_2 + L_3$, and $C \cap W = \emptyset$. Therefore, as in 5) we obtain that the blowing down $\bar{\pi}$ increases (\bar{R}^2) on $4c_{ps}$.

The neighbourhood \bar{W}' consists of $(N_2 - 2)$ components isomorphic to \bar{V}_1 , for each of which $\bar{\pi}$ is represented on Fig. 13. As above for (\bar{R}_1^2) , we see that the blowing down $\bar{\pi}$ increases (\bar{C}^2) on $2(N_2 - 2)$ for one point b and on $2(N_2 - 2)c_{ps}$ for all points of this type.

7) For a pp-point b of type A_2 the neighbourhood \bar{W} consists of two components: one is the same as in 5) and the other is the same as \bar{V}_1' and represented on the left side of Fig. 13. Since in the neighbourhood W , represented on Fig. 8, $R = R_3 + L_3$, and $C = L_2$, we obtain that the blowing down $\bar{\pi} : \bar{W} \rightarrow \tilde{W} \simeq W$ increases (\bar{R}^2) on $1 + 2 = 3$ for one point b and on $3c_{pp}$ for all points of this type. Besides, (\bar{C}^2) is increased on c_{pp} .

The neighbourhood \bar{W}' consists of $(N_2 - 3)$ components isomorphic to \bar{V}_1' , and is represented

on Fig. 13 (on which C stands for R). Therefore, taking account of the neighbourhood \bar{W} , the blowing down $\bar{\pi}$ increases (\bar{C}^2) on $2(N_2 - 3)c_{pp} + c_{pp} = (2N_2 - 5)c_{pp}$.

Summing all modifications of (\bar{R}^2) and (\bar{C}^2) , we obtain

$$(\tilde{R}^2) = (\bar{R}^2) + 2n_{sp} + 4c_{sp} + 4c_{ps} + 3c_{pp}, \quad (5.8)$$

$$(\tilde{C}^2) = (\bar{C}^2) + 2n_{sp} + (N_2 - 2)c_{sp} + 2(N_2 - 2)c_{ps} + (2N_2 - 5)c_{pp}. \quad (5.9)$$

Applying (5.5) and substituting (\bar{R}_1^2) from (5.7), we obtain

$$\begin{aligned} (\tilde{R}^2) &= 2((\tilde{R}_1^2) - c_{sp} - 2c_{ps} - 2c_{pp}) + 2n_{sp} + 4c_{sp} + 4c_{ps} + 3c_{pp} = \\ &= 2(\tilde{R}_1^2) + 2n_{sp} + 2c_{sp} - c_{pp}, \end{aligned} \quad (5.10)$$

$$\begin{aligned} (\tilde{C}^2) &= (N_2 - 2)((\tilde{R}_1^2) - c_{sp} - 2c_{ps} - 2c_{pp}) + 2n_{sp} + (N_2 - 2)c_{sp} + 2(N_2 - 2)c_{ps} + (2N_2 - 5)c_{pp} = \\ &= (N_2 - 2)(\tilde{R}_1^2) + 2n_{sp} - c_{pp}. \end{aligned} \quad (5.11)$$

5.5. Computation of intersection numbers (continuation). Now we find $(\tilde{R} + Z_R)^2$, $(\tilde{C} + Z_C)^2$ and $(\tilde{R} + Z_R) \cdot (\tilde{C} + Z_C)$, where the divisor Z_R , respectively Z_C , equals to $\sum Z_x$, where Z_x is the canonical cycle of a point $x \in \text{Sing } X$, and the summation runs over $x \in R$, respectively $x \in C$. The analogous sums $\sum \delta_x$ we denote by δ_R and δ_C respectively. By (4.2) we have

$$(\tilde{R} \cdot Z_R) = -(Z_R^2) = 2\delta_R, (\tilde{C} \cdot Z_C) = -(Z_C^2) = 2\delta_C.$$

Obviously,

$$(\tilde{R} + Z_R) \cdot (\tilde{C} + Z_C) = \tilde{R} \cdot \tilde{C}, \quad (5.12)$$

and

$$(\tilde{R} + Z_R)^2 = (\tilde{R}^2) + 2(\tilde{R} \cdot Z_R) + (Z_R^2) = (\tilde{R}^2) + 2\delta_R. \quad (5.13)$$

Analogously, $(\tilde{R}_C + Z_C)^2 = (\tilde{C}^2) + 2\delta_C$.

It remains to determine how many singular points $x \in \text{Sing } X$ lie on R , respectively, on C . From 5.2 it follows that over each ss-point on R there lie 2, and on C there lie $(N_2 - 2)$ singular points. There are no other singular points on R . There are singular points on C of the following type: over a sp-point of type A_1 there are $(N_2 - 4)$ singular points of type A_1 , over a sp-point of type A_2 there are $(N_2 - 3)$ singular points of type A_2 . We obtain

$$\delta_R = 2(\delta_0 + n_{ss} + c_{ss}) = 2(\delta_1 - n_{sp} - c_{sp}), \quad (5.14)$$

$$\begin{aligned} \delta_C &= (N_2 - 2)(\delta_0 + n_{ss} + c_{ss}) + (N_2 - 4)n_{sp} + (N_2 - 3)c_{sp} = \\ &= (N_2 - 2)\delta_1 - 2n_{sp} - c_{sp}. \end{aligned}$$

Substituting (\tilde{R}^2) from (5.10) and δ_R from (5.14) to (5.13), we obtain

$$\begin{aligned} (\tilde{R} + Z_R)^2 &= 2(\tilde{R}_1^2) + 2n_{sp} + 2c_{sp} - c_{pp} + 4(\delta_1 - n_{sp} - c_{sp}) = \\ &= 2((\tilde{R}_1^2) + 2\delta_1) - 2n_{sp} - 2c_{sp} - c_{pp}. \end{aligned}$$

Analogously we find

$$\begin{aligned} (\tilde{R}_C + Z_C)^2 &= (N_2 - 2)(\tilde{R}_1^2) + 2n_{sp} - c_{pp} + 2(N_2 - 2)\delta_1 - 4n_{sp} - 2c_{sp} = \\ &= (N_2 - 2)((\tilde{R}_1^2) + 2\delta_1) - 2n_{sp} - 2c_{sp} - c_{pp}. \end{aligned}$$

Set

$$2n_{sp} + 2c_{sp} + c_{pp} = \iota_1, \quad (5.15)$$

and let

$$g_1 = p_a(R_1) = g + \delta_1 \quad (5.16)$$

be the arithmetic genus of the curve R_1 . Since by (5.6) $(\tilde{R}_1^2) + 2\delta_1 = 3\bar{d} + g - 1 + \delta_1 = 3\bar{d} + g_1 - 1$, finally we obtain:

$$(\tilde{R} + Z_R)^2 = 2(3\bar{d} + g_1 - 1) - \iota_1, \quad (\tilde{C} + Z_C)^2 = (N_2 - 2)(3\bar{d} + g_1 - 1) - \iota_1, \quad (\tilde{R} + Z_R) \cdot (\tilde{C} + Z_C) = \iota_1. \quad (5.17)$$

5.7. *The self-intersection number of the divisor $\tilde{R} + Z_R$ is positive.*

Lemma 5.1

$$(\tilde{R} + Z_R)^2 > 0. \quad (5.18)$$

Proof. Recall that $2\bar{d} = d = \deg B$, and $\delta_1 = \delta_0 + n_{sp} + c_{sp}$. Therefore,

$$\begin{aligned} (\tilde{R} + Z_R)^2 &= 2(3\bar{d} + g - 1 + \delta_1) - 2n_{sp} - 2c_{sp} - c_{pp} = \\ &= d + (2d + 2g - 2) + 2\delta_0 - c_{pp}. \end{aligned} \quad (5.19)$$

Now we apply the Hurwitz formula for a generic projection $\varphi : B \rightarrow \mathbb{P}^1$ of the curve B from a point $P \in \mathbb{P}^2$ onto the line \mathbb{P}^1 , more precisely for the covering $\bar{\varphi} : \bar{B} \rightarrow \mathbb{P}^1$, where $\bar{\varphi} = \varphi \circ n$, and $n : \bar{B} \rightarrow B$ is a normalization of the curve B . Obviously, the covering $\bar{\varphi}$ is ramified at the following points. Firstly, $\bar{\varphi}$ has a ramification of the second order at points $\bar{b} \in \bar{B}$, which correspond to non-singular points $b \in B$, for which the line \overline{Pb} is tangent to B . The number of such points is $\hat{d} = \deg \hat{B}$, where \hat{B} is a curve dual to B . Secondly, $\bar{\varphi}$ has a ramification of order m_k at points \bar{b} , which correspond to the branches B_k of the curve B at the singular points b . Here m_k is the multiplicity (order) of the corresponding branch. Denote by

$$\nu = \sum_k (m_k - 1), \quad (5.20)$$

where the summation runs over all branches of the curve (at singular points). The covering $\bar{\varphi}$ is of degree $d = \deg B$. By the Hurwitz formula we obtain

$$2g - 2 = -2d + \hat{d} + \nu. \quad (5.21)$$

Remark 5.1 *Actually we derived one of the Plücker formulae*

$$\hat{d} = 2d + (2g - 2) - \nu$$

for a plane curve with singularities.

Obviously, the number ν for A-D-E-singularities is equal:

$$\nu(A_{2k-1}) = \nu(D_{2k+2}) = 0, \quad \nu(A_{2k}) = \nu(D_{2k+3}) = \nu(E_7) = 1, \quad \nu(E_6) = \nu(E_8) = 2.$$

Therefore, for the curve B the number $\nu = \nu(B)$ is equal:

$$\nu = c + \nu', \quad \text{where } \nu' = \sum_{k>1} a_{2k} + \sum d_{2k+3} + 2e_6 + e_7 + 2e_8. \quad (5.22)$$

Returning to the proof of the inequality, we obtain from (5.19), (5.21) and (5.22)

$$(\tilde{R} + Z_R)^2 = d + (\hat{d} + \nu) + 2\delta_0 - c_{pp} = d + \hat{d} + 2\delta_0 + \nu' + (c - c_{pp}) > 0. \quad \blacksquare \quad (5.23)$$

5.8. Conclusion of the main inequality. Applying the Hodge index theorem to divisors $\tilde{R} + Z_R$ and $\tilde{C} + Z_C$ on the surface S , we obtain

$$\left| \begin{array}{cc} 2(3\bar{d} + g_1 - 1) - \iota_1 & \iota_1 \\ \iota_1 & (N_2 - 2)(3\bar{d} + g_1 - 1) - \iota_1 \end{array} \right| \leq 0.$$

Therefore,

$$2(N_2 - 2)(3\bar{d} + g_1 - 1)^2 - N_2(3\bar{d} + g_1 - 1)\iota_1 \leq 0$$

or

$$N_2[2(3\bar{d} + g_1 - 1) - \iota_1] \leq 4(3\bar{d} + g_1 - 1). \quad (5.24)$$

Thus, if there are two nonequivalent generic coverings f_1 and f_2 , then

$$N_2 \leq \frac{4(3\bar{d} + g_1 - 1)}{2(3\bar{d} + g_1 - 1) - \iota_1}. \quad (5.25)$$

6 Proof of the Chisini conjecture for pluricanonical embeddings of surfaces of general type.

6.1. The numerical invariants in the case of a m -canonical embedding. Let S be a minimal model of a surface of general type with numerical invariants $(K_S^2) = k$ and $e(S) = e$. Let X be a canonical model of the surface S , and $\pi : S \rightarrow X$ be the blowing down of (-2) -curves. Let $f : X \rightarrow \mathbb{P}^2$ be a generic m -canonical covering, i.e. a generic projection onto \mathbb{P}^2 of $X = \varphi_m(S)$, where φ_m is a m -canonical map, $\varphi_m : S \rightarrow \mathbb{P}^{p_m-1}$, defined by the complete linear system $|mK_S|$, $p_m = \frac{1}{2}m(m-1)k + \chi(S)$. As is well known [BPV], by a theorem of Bombieri $\varphi_m(S) \simeq X$ for $m \geq 5$, and φ_m gives the blowing down π .

Let $B \subset \mathbb{P}^2$ be the discriminant curve. We conserve the notations of §4. Then

$$\bar{L} = mK_S, \quad K_S \cdot Z = 0, \quad \bar{R} = (3m + 1)K_S - Z. \quad (6.1)$$

By formulae (4.3), we obtain

$$N = m^2k, \quad d = m(3m + 1)k. \quad (6.2)$$

By formulae (4.10), we find

$$3\bar{d} + p_a(R) - 1 = (3m + 1)^2 k, \quad (6.3)$$

and

$$p_a(R) - 1 = \frac{1}{2}(3m + 1)(3m + 2)k. \quad (6.4)$$

6.2. *Invariants of a surface and of the discriminant curve define invariants of the covering.* Now let S_1 and S_2 be two surfaces of general type with numerical invariants k and e . Let $f_i : X_i \rightarrow \mathbb{P}^2$, $i = 1, 2$, be their m_i -canonical coverings having the same discriminant curve $B \subset \mathbf{P}^2$. Then by the second formula of (6.2) it follows that $m_1 = m_2 = m$. Then also $\deg f_1 = \deg f_2 = N$. We show that the other numerical invariants of f_1 and f_2 are the same.

By formula (6.4) it follows that $p_a(R_1) = p_a(R_2)$, and since $p_a(R) = g + \delta_X$, we have $\delta_{X_1} = \delta_{X_2}$.

By formulae (4.14) and (4.11) it follows that the number of p-cusps c_p and the number of p-nodes n_p for both coverings are the same. Then $n_p = n_{pp} + n_{ps} = n_{pp} + n_{sp}$, $c_p = c_{pp} + c_{ps} = c_{pp} + c_{sp}$ and, consequently, $n_{ps} = n_{sp}$ and $c_{ps} = c_{sp}$.

6.3. *The main inequality in the case of surfaces of general type.* To prove that m-canonical projections f_1 and f_2 are equivalent, by (5.24) it is sufficient to show that an inequality

$$N(2(3\bar{d} + p_a(R) - 1) - \iota) > 4(3\bar{d} + p_a(R) - 1)$$

holds (here R stands for R_1), or

$$(N - 2)(3d + 2p_a(R) - 2) - N \cdot \iota > 0, \quad (6.6)$$

where

$$\iota = 2n_{sp} + 2c_{sp} + c_{pp} = 2n_{sp} + c_{sp} + c_p. \quad (6.7)$$

Let us obtain an estimate for the number ι . We can express c_p by formulae (4.14)

$$c_p = 3N + 2p_a(R) - 2 - e. \quad (6.8)$$

To estimate $2n_{sp} + c_{sp}$ we use the Hirzebruch-Miyaoka inequality ([BPV], p.215): if the minimal surface of general type S contains s disjoint (-2) -curves, then

$$s \leq \frac{2}{9}(3e(S) - (K_S^2)). \quad (6.9)$$

Since we can take one (-2) -curve for each of the singular points of types A_1 and A_2 on X , we have

$$n_s + c_s \leq \frac{2}{9}(3e - k). \quad (6.10)$$

Remark 6.1 *Instead of the Hirzebruch-Miyaoka inequality we can use the estimate $2n_{sp} + c_{sp} \leq 2(h^{1,1} - 1) = 2(e - 2 + 4q - 2p_g - 1)$ and the inequalities $p_g \geq q$, $p_g \leq \frac{1}{2}(K_S^2) + 2$ (the Noether's inequality).*

By (6.7), (6.8) and (6.10), we obtain an estimate

$$\begin{aligned}\iota &\leq \frac{4}{9}(3e - k) + 3N + 2p_a(R) - 2 - e = \\ &= \frac{1}{3}e - \frac{4}{9}k + 3N + 2p_a(R) - 2.\end{aligned}$$

Applying the Noether's inequality ([BPV] ,p.211),

$$e \leq 5k + 36, \quad (6.11)$$

we obtain

$$\iota \leq \frac{11}{9}k + 12 + 3N + 2p_a(R) - 2. \quad (6.12)$$

Combining (6.12) and (6.6), we obtain a corollary.

Lemma 6.1 *If the inequality*

$$3N(d - N) - 6d - 4(p_a(R) - 1) - \left(\frac{11}{9}k + 12\right)N > 0 \quad (6.13)$$

holds, then a generic m -canonical projection of a surface of general type S with given k and e is unique. ■

6.4. Proof of Theorem 0.3 . Express the inequality (6.13) in terms of m . Substitute N and d from (6.2) and $p_a(R) - 1$ from (6.4) to (6.13). We obtain

$$3m^3(2m + 1)k^2 - 6m(3m + 1)k - 2(3m + 1)(3m + 2)k - \left(\frac{11}{9}k + 12\right)km^2 > 0,$$

i.e.

$$3m^3(2m + 1)k - 4(3m + 1)^2 - \left(\frac{11}{9}k + 12\right)m^2 > 0.$$

Dividing by m^2 , we obtain

$$3m(2m + 1)k - \left(\frac{11}{9}k + 12\right) - 4\left(3 + \frac{1}{m}\right)^2 > 0,$$

or, dividing by k ,

$$3m(2m + 1) > \frac{11}{9} + \frac{1}{k} \left(12 + 4\left(3 + \frac{1}{m}\right)^2\right). \quad (6.14)$$

The right side of inequality decreases, when k and m increase. This inequality holds for all $k \in \mathbf{N}$, if it holds for $k = 1$. For $k = 1$ and $m = 3$ the right side equals $\frac{11}{9} + 12 + 4 \cdot \left(\frac{10}{3}\right)^2 = \frac{173}{3} < 9 \cdot 7 = 63$. Thus, the inequality (6.14), and, consequently, the inequality (6.6), holds for $m \geq 3$ and for all k . This completes the proof of Theorem 0.3 .

We can mention in addition that for $m = 2$ the inequality (6.14) holds, if $k > 2$, and for $m = 1$ it holds, if $k > 9$.

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